ON p-ADIC PROPERTIES OF SOME FIBONACCI AND LUCAS NUMBERS

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Received: 7/19/18, Revised: 9/16/19, Accepted: 10/30/19, Published: 12/6/19

Abstract
Let $a, n \in \mathbb{Z}^+$, $b \in \mathbb{N}$, and p be a prime with $\gcd(a, p) = 1$. We consider the subsequences $\{F(ap^n + b)\}_{n \geq 0}$ and $\{L(ap^n + b)\}_{n \geq 0}$ of Fibonacci $F(n)$ and Lucas $L(n)$ numbers from the point of view of p-adic convergence. We establish the p-adic limits, the rate of convergence, and show how to calculate the limits of $F(ap^n + b)$ and $L(ap^n + b)$ if they exist.

1. Introduction

In this paper our main goal is to consider subsequences $\{F(ap^n + b)\}_{n \geq 0}$ and $\{L(ap^n + b)\}_{n \geq 0}$ from the point of view of p-adic convergence. All limits are meant p-adically. In addition we assume that $b \in \mathbb{N}$ but most of the results can be easily generalized to the case with negative values of b, typically by applications of Binet's formula. We note that the study of p-adic valuation of combinatorial quantities has become an increasingly popular subject in recent years, see e.g., [1]–[2], [5], [8], [11]–[12], and [15]–[20]. Determining p-adic convergence (cf. [10]) and effectively calculating the p-adic limits of combinatorial sequences or subsequences, provided that the limits exist, raise new questions and require new approaches.

The p-adic order, $\nu_p(a)$, of a is the exponent of the highest power of the prime p which divides the integer a. The smallest positive index n such that $F(n) \equiv 0 \pmod{p}$ is called the rank of apparition or Fibonacci entry-point of prime p and is denoted by $\rho(p)$. The order of p in $F(\rho(p))$ is denoted by $e(p) = \nu_p(F(\rho(p))) \geq 1$. We denote the modulo m period of the Fibonacci sequence by $\pi(m)$. It is known that $\rho(p^n) = \rho(p)$ for $1 \leq n \leq e(p)$ while $\rho(p^n) = \rho(p)p^{n-e(p)}$ for $n \geq e(p)$; and thus, if $\rho(p^2) \neq \rho(p)$ then $\rho(p^n) = \rho(p)p^{n-1}, n \geq 1$. We have $\rho(2) = 3$, $\rho(5) = 5$, $p \mid F(k)$ if and only if $\rho(p) \mid k$ and if $p \neq 2, 5$ then with the Legendre symbol $\left(\frac{5}{p}\right)$ we have $\rho(p) \mid p - \left(\frac{5}{p}\right) = p \pm 1$ (cf. [13]). Also, if $\pi(p^2) \neq \pi(p)$ then $\pi(p^n) = \pi(p)p^{n-1}, n \geq 1$, and if t is the largest integer with $\pi(p^t) = \pi(p)$ then $\pi(p^n) = \pi(p)p^{n-t}$ for $n \geq t$ (cf.

[23, Theorem 5]). We will assume that $\pi(p^2) \neq \pi(p)$, i.e., $\pi(p^{n+1}) = \pi(p)p^n$. We have $\pi(2) = 3$, $\pi(5) = 20$, and that $\pi(p)/\rho(p)$ is 1, 2, or 4. The latter case implies that $p \equiv 1 \pmod 4$; cf. [7]. Note that if $p \neq 2, 5$, then $\pi(p) \mid p - 1$ if $\left(\frac{5}{p}\right) = 1$ and $\pi(p) \mid 2(p+1)$ if $\left(\frac{5}{p}\right) = -1$ (cf. [23, Theorems 6 and 7]), and by the quadratic reciprocity theorem, we have that $\left(\frac{5}{p}\right) = 1$ and $\left(\frac{5}{p}\right) = -1$ exactly if $p \equiv \pm 1 \pmod{10}$ and $p \equiv \pm 3 \pmod{10}$, respectively. The notion of the rank of apparition and period can be easily generalized for any integer $m \geq 1$. Note that the modulo m periods $\pi(m)$ and $\pi_L(m)$ for the Fibonacci and Lucas numbers, respectively, can be different only if $\gcd(m, 5) \neq 1$. The reason is that $L(n) = F(n+1) + F(n-1)$ and $5F(n+1) = 3L(n) + L(n-1)$, $n \geq 1$, are expressed as linear combinations of the Fibonacci and Lucas numbers, respectively, with integer coefficients. In fact, $\pi_L(5^n)$ divides $\pi(5^n)$ since $\pi(5^n) = 5^n \cdot 4$ and $\pi_L(5^n) = 5^{n-1} \cdot 4$, $n \geq 1$.

We also use the notation $\mathrm{inv}(n, m)$ for the modulo m inverse of n if it exists.

We proved the following theorem on $\nu_p(F(n))$ and $\nu_p(L(n))$ in [6].

Theorem 1.1 ([7, Theorem A] and [6, Lemma 2 and Theorem]). *For all $n \geq 0$ and prime p we have $\nu_5(F(n)) = \nu_5(n)$ and*

$$\nu_2(F(n)) = \begin{cases} 0, & \text{if } n \equiv 1, 2 \pmod 3, \\ 1, & \text{if } n \equiv 3 \pmod 6, \\ 3, & \text{if } n \equiv 6 \pmod{12}, \\ \nu_2(n) + 2, & \text{if } n \equiv 0 \pmod{12}. \end{cases} \tag{1.1}$$

If $p \neq 2, 5$ then

$$\nu_p(F(n)) = \begin{cases} \nu_p(n) + e(p), & \text{if } n \equiv 0 \pmod{\rho(p)}, \\ 0, & \text{otherwise.} \end{cases} \tag{1.2}$$

For the Lucas numbers we have $\nu_5(L(n)) = 0$,

$$\nu_2(L(n)) = \begin{cases} 0, & \text{if } n \equiv 1, 2 \pmod 3, \\ 2, & \text{if } n \equiv 3 \pmod 6, \\ 1, & \text{if } n \equiv 0 \pmod 6, \end{cases} \tag{1.3}$$

and if $p \neq 2, 5$ then

$$\nu_p(L(n)) = \begin{cases} \nu_p(n) + e(p), & \text{if } \pi(p) \neq 4\rho(p) \text{ and } n \equiv \frac{\rho(p)}{2} \pmod{\rho(p)}, \\ 0, & \text{otherwise.} \end{cases} \tag{1.4}$$

We know that the rank of apparition $\rho(m)$ is odd exactly if $\pi(m) = 4\rho(m)$; cf. [22, Theorem 1]. Note that in [9] we discussed the p-adic order and related congruences for the differences $L(ap^{n+1}) - L(ap^n)$.

We also have various applications of some of the standard addition formulas for Fibonacci and Lucas numbers, cf. (2.3) and (2.4), as well as the duplication identity (2.2).

In most cases when necessary we assume that $\gcd(a, p) = 1$. We focus on the cases with $p = 2$ and $p = 5$ in Sections 2 and 3, respectively. The general case is treated in Section 4. Section 5 contains the proofs and Section 6 is devoted to some examples.

The p-adic limits are established in Theorems 2.2, 2.4, 2.5, 2.14, 3.2, 3.3, 4.1, and 4.3. Regarding the rate of convergence we have Theorems 2.2, 2.11, 3.6, 4.1, 4.3, 4.12, 4.14, and Lemmas 2.7, 2.9, 3.4, 4.8, and 4.9, providing exact rates or lower bounds on them. Some of the limits are determined by or related to each other in Remark 4.4, Theorems 2.13, 3.3, 3.5, 4.5, 4.7, 4.13, Lemmas 4.8 and 4.9. Remark 4.4 and Theorem 4.17 list cases when the limits, except for $\lim_{n\to\infty} L(ap^n)$, do not exist. Some lemmas of independent interest on $L(ap^n) - 2$ (and $L(ap^n) - 2\left(\frac{5}{p}\right)$) modulo a high power of p and on its p-adic order are provided in Lemmas 2.7, 2.9, 3.4, 4.8, and 4.9.

Remark 1.2. We note that Rowland and Yassawi addressed some general issues concerning the p-adic asymptotic analysis of general linear recurrences with constant coefficients in [15], and some of the results in this paper regarding the calculations of the limits of Fibonacci sequences fall under the scope of [15]. However, their methods are different and the results do not address some of our questions, the rate of convergence, in particular. We also include results on Lucas numbers whose behavior is different from that of the Fibonacci numbers. Although the approach in [15] can be applied to the Lucas numbers but it was left to the reader; and thus, these differences were not addressed. Our main goal is to establish the existence of various kinds of limits, their interplay, and ways of calculating the limits by gaining their extra p-adic digits.

Rowland and Yassawi used sophisticated p-adic analytic techniques in [15, Corollary 11] and found that the limits $\lim_{n\to\infty} F(2^{2n})$ and $\lim_{n\to\infty} F(2^{2n+1})$ are equal to $-\sqrt{-3/5}$ and $\sqrt{-3/5}$, respectively. The corollary can be used to calculate limits of the form $\lim_{n\to\infty} F(ap^{fn} + b)$, p prime, $a, b \in \mathbb{Z}$ with $a \geq 1$ and $f = f(p) = 1$ or 2, in general. The value of the limit is algebraic over $\mathbb{Q}_p$. For instance, $\lim_{n\to\infty} F(3^{2n}) = \sqrt{2/5}$ and $\lim_{n\to\infty} F(3^{2n+1}) = -\sqrt{2/5}$, respectively. Also,

$\lim_{n\to\infty} F(11^n)$ is the root of $5x^2 + 5x + 1$. In fact, it is

$$-\frac{1}{2} + \frac{1}{2\sqrt{5}},$$

which does not immediately follow from the corollary. The corollary also implies Theorem 2.14 which completely characterizes the 2-adic limit $\lim_{n\to\infty} F(a2^{2n} + b)$. Other possibilities are discussed in [15, Corollary 12], e.g., sufficient conditions are given for $\lim_{n\to\infty} F(ap^{fn} + b) = F(b)$. Our Theorem 2.5 corresponds to this result for $p = 2$ and Theorem 2.11 extends it by establishing the exact rate of convergence. For general primes Theorems 4.13 and 4.14 achieve the same (or at least provide a lower bound on the rate of convergence).

2. The Case of $p = 2$

Jacobson proved for $n \geq 5$ and $a \geq 0$ that

$$F(2^{n-3} \cdot 3a) \equiv a2^{n-1} \pmod{2^n},$$

$$F(2^{n-3} \cdot 3a \pm b) \equiv F(b) \pmod{2^n}, \text{ if } b \equiv 3 \pmod{6},$$
$$F(2^{n-3} \cdot 3 + b) \equiv F(b) + 2^{n-1} \pmod{2^n}, \text{ if } b \equiv 0 \pmod{6},$$

and

$$F(2^{n-3} \cdot 3 + b) \equiv F(b) \pmod{2^{n-1}}, \text{ if } b \equiv 0 \pmod{6}$$

in [4, Lemmas 2–4, and 6]. This guarantees that $\lim_{n\to\infty} F(2^n \cdot 3a) = 0$, $\lim_{n\to\infty} F(2^n \cdot 3 + b) = F(b)$ with $b \equiv 0 \pmod{6}$, and $\lim_{n\to\infty} F(2^n \cdot 3a \pm b) = F(b)$ with $b \equiv 3 \pmod{6}$ exist in the 2-adic sense.

We use the following lemma in the proofs.

Lemma 2.1 ([4, Lemma 3]). *Let $n \geq 2$ and $a \geq 1$. Then,*

$$F(2^n \cdot 3a - 1) \equiv 1 - a2^{n+1} \pmod{2^{n+3}},$$

and

$$F(2^n \cdot 3a) \equiv a2^{n+2} \pmod{2^{n+3}}.$$

The following theorem establishes the existence of limits and in Theorem 2.4 we reduce the study to finding $\lim_{n\to\infty} F(a2^n + b)$ and $\lim_{n\to\infty} F(2^n \cdot 3a + b)$. We prove that the latter limit equals $F(b)$ in Theorem 2.5. This implies that $\lim_{n\to\infty} F(2^n \cdot 3a) = 0$.

Theorem 2.2. *We have that*

$$F(a2^{n+2}+b) \equiv F(a2^n+b) \pmod{2^{n+1}}$$

and

$$L(a2^{n+2}+b) \equiv L(a2^n+b) \pmod{2^{n+1}}.$$

These congruences already guarantee the existence of the 2-adic limits $\lim_{n\to\infty} F(a2^{2n}+b)$, $\lim_{n\to\infty} F(a2^{2n+1}+b)$, $\lim_{n\to\infty} L(a2^{2n}+b)$, $\lim_{n\to\infty} L(a2^{2n+1}+b)$, *and lower bounds on the rate of convergence*

$$\nu_2(F(a2^{n+2}+b) - F(a2^n+b)) \geq n+1$$

and

$$\nu_2(L(a2^{n+2}+b) - L(a2^n+b)) \geq n+1.$$

Remark 2.3. We notice that if $b = 0$ and $a \not\equiv 0 \pmod 3$ then $\lim_{n\to\infty} F(a2^{2n} + b) \neq \lim_{n\to\infty} F(a2^{2n+1}+b)$ while their sum is the 2-adic 0. This possibility is further explored in Remark 2.8, Theorems 2.14 and 4.7 (for a general prime p), in general.

A similar proof will be presented for Theorem 4.1, too.

Now we explore how to calculate some of these limits. The following Theorem 2.4 helps to reduce the problem of finding $\lim_{n\to\infty} F(a2^n+b)$ to $\lim_{n\to\infty} F(a'2^n+b)$ with $a' = 0, 1, 2$. Theorem 2.5 takes care of the first case. Beyond that, we have to deal with the case of $a' = 1$ only since $a' = 2$ reduces to this case. In this case, we give only a method for calculating the limits. Of course, Theorem 2.2 already suggests that we get more and more 2-adic digits of the limits as $2n$ and $2n+1$ increase, respectively. In Theorem 2.11 we exhibit the exact rate of convergence of the sequences $F(2^n \cdot 3a + b)$ and $L(2^n \cdot 3a + b)$ in (2.5) and (2.6), as well as those of $F(a'2^{2n}+b)$, $F(a'2^{2n+1}+b)$, $L(a'2^{2n}+b)$, and $L(a'2^{2n+1}+b)$ in (2.7) and (2.8) for $a' \equiv 1 \pmod 3$ provided that $b = 0$.

Theorem 2.4. *Let $n \geq 2$ and a' denote the modulo 3 remainder of a. We have that*

$$F(a2^n+b) \equiv F(a'2^n+b) \pmod{2^{n+1}}$$

and

$$L(a2^n+b) \equiv L(a'2^n+b) \pmod{2^{n+1}}.$$

In particular, we have the following case.

Theorem 2.5. *Let $n \geq 2$, $a \geq 1$, and b be an integer. We have that*

$$F(2^n \cdot 3a + b) \equiv F(b) \pmod{2^{n+1}},$$

and

$$L(2^n \cdot 3a + b) \equiv L(b) \pmod{2^{n+1}}.$$

As far as Lucas numbers are concerned we recall a lemma.

Lemma 2.6 ([7, Lemma 1']). *For $n \geq 0$ we have that*

$$L(2^n \cdot 12) \equiv 2 \pmod{2^{2n+6}}.$$

It follows that 2-adically $\lim_{n\to\infty} L(2^n \cdot 12) = 2$. We can extend this lemma and see that $\lim_{n\to\infty} L(a2^n)$ always exists.

Lemma 2.7. *For $n \geq 0$ and $a \geq 1$ we have that*

$$L(2^n \cdot 12a) \equiv 2 \pmod{2^n}$$

and

$$L(a2^n) \equiv -1 \pmod{2^{n+1}}, \quad \text{if } a \not\equiv 0 \pmod 3,$$

which gives the 2-adic limits $\lim_{n\to\infty} L(a2^n) = 2$ *if* $3 \mid a$ *and* $\lim_{n\to\infty} L(a2^n) = -1$ *if* $3 \nmid a$. *In the latter case, with a odd and $n \geq 1$, we also have that*

$$\nu_2(L(a2^n) + 1) = n + 1. \tag{2.1}$$

Remark 2.8. The limit $\lim_{n\to\infty} L(a2^n) = -1$ combined with Theorem 2.2 yields that $\lim_{n\to\infty} F(a2^{2n}) + \lim_{n\to\infty} F(a2^{2n+1}) = 0$ by the duplication identity (cf. [13, (IV.2)])

$$F(2n) = F(n)L(n). \tag{2.2}$$

This observation can be generalized, cf. Remark 2.3.

In a similar fashion to Lemma 2.6 we can improve Lemma 2.7.

Lemma 2.9. *For $n \geq 0$ and $a \geq 1$ odd we have that*

$$L(2^n \cdot 12a) \equiv 2 \pmod{2^{2n+6}}$$

and

$$\nu_2(L(2^n \cdot 12a) - 2) = 2n + 6.$$

For the rate of 2-adic convergence we obtain the following theorem. We use some standard identities for the Fibonacci and Lucas numbers. Two identities have fairly similar forms (cf. [13, (IV.7) and (IV.4)]):

$$2F(m + m') = F(m)L(m') + F(m')L(m) \tag{2.3}$$

and

$$2L(m + m') = L(m)L(m') + 5F(m')F(m) \tag{2.4}$$

with nonnegative integers m and m'. These identities remain true for negative integers too by Binet's formula.

Remark 2.10. The summation identities (2.3) and the generalized version of (2.4) might play a role in proving general results similar to the following ones for other Lucas and their companion sequences [13, Section IV of Chapter 2].

Theorem 2.11. *If $n \geq 3$ and $a \geq 1$ odd then*

$$\nu_2(F(2^{n+1} \cdot 3a + b) - F(2^n \cdot 3a + b)) = n + 1 + \nu_2(L(b)), \tag{2.5}$$

$$\nu_2(L(2^{n+1} \cdot 3a+b) - L(2^n \cdot 3a+b)) = \begin{cases} 2n+2, & \text{if } b = 0, \\ n + 1 + \nu_2(F(b)), & \text{if } b \neq 0 \text{ and } n > \nu_2(F(b)), \end{cases} \tag{2.6}$$

$$\nu_2(F(a2^{n+2}) - F(a2^n)) = n + 1, \quad \text{if } a \not\equiv 0 \pmod 3, \tag{2.7}$$

and

$$\nu_2(L(a2^{n+2}) - L(a2^n)) = n + 1, \quad \text{if } a \not\equiv 0 \pmod 3. \tag{2.8}$$

Moreover, if $n \geq 6$ then with $a' \equiv a \not\equiv 0 \pmod 3$ we get that

$$f(n) = \nu_2(F(a2^{n+2} + b) - F(a2^n + b)) = n + 1 + \nu_2(L(4a'(-1)^n + b)) \tag{2.9}$$

and

$$\begin{aligned} l(n) &= \nu_2(L(a2^{n+2} + b) - L(a2^n + b)) \\ &= \begin{cases} n + 1 + \nu_2(F(b)) + 2, & \text{if } a2^n + b \equiv 0 \pmod{12} \text{ and } n > \nu_2(b) + 2, \\ n + 1 + \nu_2(F(4a'(-1)^n + b)), & \text{if } a2^n + b \not\equiv 0 \pmod{12}. \end{cases} \end{aligned} \tag{2.10}$$

where $\nu_2(L(b))$ and $\nu_2(F(b))$ are determined in Theorem 1.1. For any sufficiently large n we have

$$f(n+2) = f(n) + 2 \text{ and } l(n+2) = l(n) + 2. \tag{2.11}$$

Remark 2.12. Identities (2.9) and (2.10) are generalizations of (2.7) and (2.8) by including the term b.

We will see in Theorem 4.17 that the limits $\lim_{n\to\infty} F(a2^n + b)$ and $\lim_{n\to\infty} L(a2^n + b)$ exist exactly if $a \equiv 0 \pmod 3$, otherwise the sequences split toward two limit values, for both the Fibonacci and Lucas numbers: $\lim_{n\to\infty} F(a2^{2n} + b)$ vs. $\lim_{n\to\infty} F(a2^{2n+1} + b)$ and $\lim_{n\to\infty} L(a2^{2n+1} + b)$ vs. $\lim_{n\to\infty} L(a2^{2n} + b)$. Theorem 2.13 gives further details.

Theorem 2.13. *If $a \not\equiv 0 \pmod 3$ then $\lim_{n\to\infty} F(a2^{2n}) + \lim_{n\to\infty} F(a2^{2n+1}) = 0$ and in general, $\lim_{n\to\infty} F(a2^{2n}+b)+\lim_{n\to\infty} F(a2^{2n+1}+b) = -F(b)$. In addition, if $b \neq 0$ then neither $\lim_{n\to\infty} L(a2^n + b)$ nor $\lim_{n\to\infty}(L(a2^n + b))^2$ exists.*

By applying [15, Corollary 11] we can improve Theorem 2.13.

Theorem 2.14. *For $a \in \mathbb{Z}^+$ and $b \in \mathbb{N}$ we have the 2-adic limit*

$$\lim_{n\to\infty} F(a2^{2n} + b) = \begin{cases} F(b), & \text{if } a \equiv 0 \pmod 3, \\ -\frac{1}{2}F(b) - \frac{1}{2}\sqrt{-\frac{3}{5}}L(b), & \text{if } a \equiv 1 \pmod 3, \\ -\frac{1}{2}F(b) + \frac{1}{2}\sqrt{-\frac{3}{5}}L(b), & \text{if } a \equiv 2 \pmod 3. \end{cases}$$

3. The Case of $p = 5$

Numerical experimentations suggest the following examples.

Example 3.1. For $p = 5$ we have

$$\lim_{n\to\infty} F(a5^n) = 0, \text{ with } \gcd(a, 5) = 1, \tag{3.1}$$

$$\lim_{n\to\infty} F(5^n \cdot 2 + b) = -b, \text{ for } b = 1, 5, \tag{3.2}$$

and

$$\lim_{n\to\infty} F(5^n \cdot 2 + b) = -(b - 1), \text{ for } b = 2, 3, 4. \tag{3.3}$$

The following theorem guarantees the existence of 5-adic limits.

Theorem 3.2. *For $a \in \mathbb{Z}^+$ and $b \in \mathbb{N}$ we have that the 5-adic limits*

$$\lim_{n\to\infty} F(a5^n + b)$$

and

$$\lim_{n\to\infty} L(a5^n + b)$$

exist.

In fact, Example 3.1 follows by the following theorem which helps to determine all limits for $p = 5$. Theorem 3.6 provides the rate of convergence.

Theorem 3.3. *Assume that $\gcd(a, 5) = 1$. We have that $\lim_{n\to\infty} F(a5^n) = 0$ and if $b \geq 1$ then*

$$\lim_{n\to\infty} F(a5^n + b) = \lim_{n\to\infty} \operatorname{inv}(2^{a5^n+b-1}, 5^n) \cdot \sum_{k\leq(b-1)/2} \binom{a5^n + b}{2k + 1} 5^k. \tag{3.4}$$

We also have that for any integer b

$$\lim_{n\to\infty} 2^{a5^n+b-1}F(a5^n+b) = 2^{b-1}F(b), \tag{3.5}$$

which implies that

$$\lim_{n\to\infty} F(a5^n+b) = F(b)\cdot \lim_{n\to\infty} \mathrm{inv}(2^{a5^n}, 5^n). \tag{3.6}$$

This limit equals $-F(b)$ *if* $a \equiv 2 \pmod 4$ *and equals* $F(b)$ *if* $a \equiv 0 \pmod 4$, *since in this case* $\mathrm{inv}(2^{a5^n}, 5^n) = -1$ *or* 1, *respectively. We also have that*

$$\lim_{n\to\infty} 2L(ap^n+b) = L(b)\cdot \lim_{n\to\infty} L(a5^n)$$

and the limit $\lim_{n\to\infty} L(a5^n+b)$ *equals* $-L(b)$ *if* $a \equiv 2 \pmod 4$ *and* $L(b)$ *if* $a \equiv 0 \pmod 4$.

An analog of Lemma 2.9 is given in Lemma 3.4 which also provides insight into $\lim_{n\to\infty} L(a5^n)$ with a even. It is further generalized in Theorem 3.5.

Lemma 3.4. *For* $n \geq 0$ *and* $a \geq 1$ *we have that*

$$L(5^n\cdot 4a) \equiv 2 \pmod{5^{2n+1}};$$

furthermore, $\nu_5(L(5^n\cdot 4a)-2) = 2n+1$ *if* $\gcd(a,5)=1$. *If* a *is odd, then*

$$L(5^n\cdot 2a) \equiv -2 \pmod{5^{2n+1}}$$

and $\nu_5(L(5^n\cdot 2a)+2) = 2n+1$ *if* $\gcd(a,5)=1$.

Theorem 3.5. *For the* 5*-adic limit* $L = \lim_{n\to\infty} L(a5^n)$ *we have that* $L^4 = 16$. *If* a *is even then* L *must be 2 or -2 depending upon whether* $a \equiv 0$ *or* $2 \pmod 4$.

Theorem 3.6. *For* $a, n \in \mathbb{Z}^+$ *with* $\gcd(a,5)=1$ *and* $b \in \mathbb{N}$ *we have that*

$$\nu_5(F(a5^{n+1}+b) - F(a5^n+b)) = n, \tag{3.7}$$

$$\nu_5(L(a5^{n+1}+b) - L(a5^n+b)) = n+1+\nu_5(b), \text{ if } b \neq 0 \text{ and } n > \nu_5(b), \tag{3.8}$$

and

$$\nu_5(L(a5^{n+1}) - L(a5^n)) = 2n+1, \text{ if } b = 0. \tag{3.9}$$

4. The General Case

We have the following theorems and facts regarding the existence of the limits, their calculations and the rates of convergence. Theorem 4.1 establishes that the limits exist by using periodicity only. We continue on with finding the rate of convergence of the related sequences in Theorem 4.12. Finally, we consider some of the limits in Theorem 4.13 and Lemmas 4.8, and 4.9.

Theorem 4.1. *For $a, n \in \mathbb{Z}^+$, $b \in \mathbb{N}$, and any prime $p \neq 2, 5$, we have that*

$$F(ap^{n+2} + b) \equiv F(ap^n + b) \pmod{p^{n+1}} \tag{4.1}$$

and

$$L(ap^{n+2} + b) \equiv L(ap^n + b) \pmod{p^{n+1}}. \tag{4.2}$$

These congruences already guarantee the existence of the p-adic limits $\lim_{n\to\infty} F(ap^{2n} + b)$, $\lim_{n\to\infty} F(ap^{2n+1} + b)$, $\lim_{n\to\infty} L(ap^{2n} + b)$, $\lim_{n\to\infty} L(ap^{2n+1} + b)$, *and lower bounds on the rates of convergence*

$$\nu_p(F(ap^{n+2} + b) - F(ap^n + b)) \geq n + 1$$

and

$$\nu_p(L(ap^{n+2} + b) - L(ap^n + b)) \geq n + 1.$$

Remark 4.2. The question remains whether the exponent $n + 2$ in ap^{n+2} can be replaced by $n + 1$ in (4.1) and (4.2). We note that if the Legendre symbol $\left(\frac{5}{p}\right) = 1$, i.e., $p \equiv \pm 1 \pmod{10}$, then $\pi(p) \mid p - 1$ which makes the difference $(ap^{n+1} + b) - (ap^n + b) = ap^n(p - 1)$ of the indices a multiple of $\pi(p^{n+1}) = \pi(p)p^n$ if $\pi(p^2) \neq \pi(p)$; therefore,

$$F(ap^{n+1} + b) - F(ap^n + b) \equiv L(ap^{n+1} + b) - L(ap^n + b) \equiv 0 \pmod{p^{n+1}},$$

and the exponent $n + 2$ can be reduced to $n + 1$. The case with $\left(\frac{5}{p}\right) = -1$ seems more complex. Theorem 4.3 states that the limit $\lim_{n\to\infty} L(ap^n)$ always exists but the case of the Fibonacci numbers is different and more complicated.

We proved Corollary 3 in [9] which claims that $L(ap^n)$ forms a Honda-sequence and thus, it implies that $\lim_{n\to\infty} L(ap^n)$ always exists.

Theorem 4.3 ([9, Theorem 4]). *For $n \geq 0$, $a \geq 1$ integer, and any prime p we have that $\nu_p(L(ap^{n+1}) - L(ap^n)) \geq n + 1$.*

Remark 4.4. Recall that if $p = 2$ and $L = \lim_{n\to\infty} L(a2^n)$ then L must be either -1 or 2 by (5.2) as we claimed in Lemma 2.7. For an odd prime p we can apply [13, (IV.16)]. It claims that $L(n) \mid L(kn)$ for $n \geq 1$ and odd $k \geq 1$, which by setting $k = p$, gives rise to the possibility of the limit $L = 0$. We derive Theorem 4.5 to establish the possible limits for $p = 3$. We obtain that $\lim_{n\to\infty} L(3^n \cdot 2) = \lim_{n\to\infty} L(3^n \cdot 6) = 0$, $\lim_{n\to\infty} L(3^n \cdot 4) = -2$, and $\lim_{n\to\infty} L(3^n \cdot 8) = 2$, etc. The applied method is generalized in Theorem 3.5 to obtain the limits for $p = 5$.

We now list some of the simple cases when $\lim_{n\to\infty} L(ap^n + b)$ does not exist and the sequence $\{L(ap^n + b)\}_{n\geq 0}$ splits toward two limits: $L(2^n + 5), L(2^n \cdot 5 + b)$ with $b \neq 0$, $L(2^n \cdot 7 + 5)$, and $L(3^n + 2), L(3^n \cdot 2 + b)$ with $b \neq 0$, 2-adically and 3-adically, respectively, etc. Similar statements apply to $F(ap^n + b)$ by Theorem 4.13.

The nonexistence of $\lim_{n\to\infty} L(3^n \cdot 2 + b)$ and $\lim_{n\to\infty} F(3^n \cdot 2 + b)$ with $b \neq 0$ follow by the addition formulas (2.3) and (2.4) since

$$2F(3^{n+1} \cdot 2 + b) = F(3^n \cdot 4)L(3^n \cdot 2 + b) + F(3^n \cdot 2 + b)L(3^n \cdot 4)$$

and

$$2L(3^{n+1} \cdot 2 + b) = L(3^n \cdot 4)L(3^n \cdot 2 + b) + 5F(3^n \cdot 2 + b)F(3^n \cdot 4),$$

and we observe that $\lim_{n\to\infty} F(3^n \cdot 4) = 0$, $\lim_{n\to\infty} F(3^n \cdot 2 + b) \neq 0$, and $\lim_{n\to\infty} L(3^n \cdot 2 + b) \neq 0$ by Theorem 1.1 and $\lim_{n\to\infty} L(3^n \cdot 4) = -2$ by Lemma 4.9.

We note that Theorems 4.13 and 4.17 describe some general cases when the limit does and does not exist, respectively.

Theorem 4.5. *For $p = 3$ and a even, we have that the 3-adic limit $\lim_{n\to\infty} L(a3^n)$ is either 0, 2, or -2. For a odd, we have $\lim_{n\to\infty}(L(a3^n))^2 = -2$.*

Example 4.6. Note that $\lim_{n\to\infty}(L(3^n \cdot 7))^2 = -2$.

A similar statement is true for $p = 5$, cf. Theorem 3.5.

Numerical calculations suggest that if $\lim_{n\to\infty} F(ap^{2n}+b) \neq \lim_{n\to\infty} F(ap^{2n+1}+b)$ then often their sum is the p-adic 0. The following theorem sheds light on this observation provided that $\lim_{n\to\infty}(L(ap^n + b))^2$ exists.

Theorem 4.7. *If the p-adic limit $L' = \lim_{n\to\infty}(L(ap^n + b))^2$ exists then for $p \neq 5$, we have that $\lim_{n\to\infty}(F(ap^{2n} + b))^2 = \lim_{n\to\infty}(F(ap^{2n+1} + b))^2 = \lim_{n\to\infty}(L' - 4(-1)^{ap^n+b}) \cdot \mathrm{inv}(5, p^n)$. This guarantees that either $\lim_{n\to\infty} F(ap^{2n} + b) = \lim_{n\to\infty} F(ap^{2n+1} + b)$ or $\lim_{n\to\infty} F(ap^{2n} + b) + \lim_{n\to\infty} F(ap^{2n+1} + b) = 0$. Both $\lim_{n\to\infty} L(ap^n + b)$ and $L' = \lim_{n\to\infty}(L(ap^n + b))^2$ exist if $b = 0$ and thus, either $\lim_{n\to\infty} F(ap^{2n}) = \lim_{n\to\infty} F(ap^{2n+1})$ or $\lim_{n\to\infty} F(ap^{2n}) + \lim_{n\to\infty} F(ap^{2n+1}) = 0$.*

We will use the congruence of the following lemma.

Lemma 4.8. *For $a, n \in \mathbb{Z}^+$ with $\gcd(a, p) = 1$ and prime $p \neq 2, 5$ we have that*

$$L(a(p^2 - 1)p^n) \equiv 2 \pmod{p^{2(n+e(p))}},$$

moreover, $\nu_p(L(a(p^2 - 1)p^n) - 2) = 2(n + e(p))$.

We also have the following lemma which is included as a curious follow-up to Lemma 4.8. It will be used in the proof of Lemma 4.15.

Lemma 4.9. *For prime $p \neq 2,5$, and $a, n \in \mathbb{Z}^+$ with $\gcd(a,p)=1$, and either $\left(\frac{5}{p}\right) = -1$, $p \equiv 3 \pmod 4$, and a is an odd multiple of $\rho(p)/2$, or simply $\left(\frac{5}{p}\right) = 1$ then we have that*

$$L(a(p-1)p^n) \equiv 2\left(\frac{5}{p}\right) \pmod{p^{2(n+e(p))}},$$

moreover, $\nu_p\left(L(a(p-1)p^n) - 2\left(\frac{5}{p}\right)\right) = 2(n+e(p))$.

If $p \neq 2,5$ prime then we also have that

$$L(a(p-1)p^n) \equiv 2 \pmod{p^{2(n+e(p))}} \tag{4.3}$$

and $\nu_p(L(a(p-1)p^n) - 2) = 2(n+e(p))$ if a is a multiple of $\rho(p)$.

Remark 4.10. Note that if $\left(\frac{5}{p}\right) = -1$ and $p \equiv 3 \pmod 4$ then $\pi(p) = 2\rho(p)$; cf. [7].

We also note the following lemma.

Lemma 4.11. *For any prime $p \neq 2,5$ and $r \in \mathbb{Z}^+$ we have*

$$L(r\rho(p)) \equiv 2\left(\frac{L(\rho(p))}{2}\right)^r \pmod{p^{2e(p)}}. \tag{4.4}$$

Therefore, we have that

$$L(r\rho(p)(p-1)p^n) \equiv 2 \pmod{p^2}$$

while for $\left(\frac{5}{p}\right) = -1$ and $p \equiv 3 \pmod 4$, we have that

$$L(r\frac{\rho(p)}{2}(p-1)p^n) \equiv 2(-1)^r \pmod{p^2}.$$

We find the rate of convergence in the next theorem.

Theorem 4.12. *For $a, n \in \mathbb{Z}^+$ with $\gcd(a,p)=1$, $b \in \mathbb{N}$, and prime $p \neq 2,5$ we have*

$$\nu_p(F(ap^{n+2}) - F(ap^n)) \begin{cases} \geq 2(n+e(p)), & \text{if } a \equiv \rho(p)/2 \pmod{\rho(p)} \text{ and } \pi(p) \neq 4\rho(p), \\ = n+e(p), & \text{otherwise;} \end{cases}$$

$$\nu_p(L(ap^{n+2}) - L(ap^n)) \begin{cases} \geq 2(n+e(p)), & \text{if } a \equiv 0 \pmod{\rho(p)}, \\ = n+e(p), & \text{otherwise;} \end{cases}$$

and if $b \neq 0$ *then*

$$\nu_p(F(ap^{n+2}+b)-F(ap^n+b)) = \begin{cases} n+2e(p)+\nu_p(b), & \text{if } ap^n+b \equiv \rho(p)/2 \pmod{\rho(p)} \\ & \text{and } \pi(p) \neq 4\rho(p) \text{ and } n > \nu_p(b), \\ n+e(p), & \text{otherwise;} \end{cases}$$

$$\nu_p(L(ap^{n+2}+b)-L(ap^n+b)) = \begin{cases} n+2e(p)+\nu_p(b), & \text{if } ap^n+b \equiv 0 \pmod{\rho(p)} \\ & \text{and } n > \nu_p(b), \\ n+e(p), & \text{otherwise.} \end{cases}$$

In the following theorem we give conditions that guarantee the existence of the limits $\lim_{n\to\infty} F(ap^n)$, $\lim_{n\to\infty} F(ap^n+b)$, and $\lim_{n\to\infty} L(ap^n+b)$. We relate them to $\lim_{n\to\infty} L(ap^n)$, which might help us to determine the actual limit values.

Theorem 4.13. *Let* $a, n \in \mathbb{Z}^+$, $b \in \mathbb{N}$, *and* p *be a prime with* $\gcd(a,p)=1$. *For* $b \neq 0$ *the limits* $\lim_{n\to\infty} F(ap^n)$, $\lim_{n\to\infty} F(ap^n+b)$, *and* $\lim_{n\to\infty} L(ap^n+b)$ *exist simultaneously. In fact, if* $\lim_{n\to\infty} F(ap^n)$ *exists then the limits* $\lim_{n\to\infty} F(ap^n+b)$ *and* $\lim_{n\to\infty} L(ap^n+b)$ *exist for any* $b \neq 0$. *For* $p=2$ *they exist if* $a \equiv 0 \pmod 3$ *while for* $p=5$ *they always exist. For* $p \neq 2,5$ *the limits exist if* $\left(\frac{5}{p}\right)=1$ *or* a *is a multiple of* $\rho(p)$. *In the latter case we have*

$$\lim_{n\to\infty} F(ap^n) = 0, \tag{4.5}$$

$\lim\limits_{n\to\infty} 2F(ap^n+b) = F(b)\cdot \lim\limits_{n\to\infty} L(ap^n)$, $\lim\limits_{n\to\infty} 2L(ap^n+b) = L(b)\cdot \lim\limits_{n\to\infty} L(ap^n)$, *and therefore,*

$$L(b)\cdot \lim_{n\to\infty} F(ap^n+b) = F(b)\cdot \lim_{n\to\infty} L(ap^n+b). \tag{4.6}$$

On the other hand, we have

$$\lim_{n\to\infty} L(ap^n) = 0 \tag{4.7}$$

if a *is an odd multiple of* $\rho(p)/2$ *and* $\pi(p) \neq 4\rho(p)$.

Note that $\pi(p) \neq 4\rho(p)$ if $\left(\frac{5}{p}\right) = -1$ and $p \equiv 3 \pmod 4$ according to Remark 4.10.

In light of Theorem 4.13 we are interested, when applicable, in the rates of convergence of $\lim_{n\to\infty} F(ap^n)$, $\lim_{n\to\infty} F(ap^n+b)$, and $\lim_{n\to\infty} L(ap^n+b)$.

Theorem 4.14. *Let $a, n \in \mathbb{Z}^+$ with $\gcd(a,p) = 1$, $b \in \mathbb{N}$, $p \neq 2, 5$ prime. We assume that either $\left(\frac{5}{p}\right) = 1$ or a is a multiple of $\rho(p)$. We have that if $b = 0$ then*

$$\nu_p(F(ap^{n+1}) - F(ap^n)) \begin{cases} \geq 2(n + e(p)), & \text{if } \left(\frac{5}{p}\right) = 1 \text{ and } a \equiv \rho(p)/2 \pmod{\rho(p)} \\ & \text{and } \pi(p) \neq 4\rho(p), \\ = n + e(p), & \text{otherwise,} \end{cases}$$

and if $b \neq 0$ then

$$\nu_p(F(ap^{n+1} + b) - F(ap^n + b)) = \begin{cases} n + 2e(p) + \nu_p(b), & \text{if } ap^n + b \equiv \rho(p)/2 \pmod{\rho(p)} \\ & \text{and } \pi(p) \neq 4\rho(p) \text{ and } n > \nu_p(b), \\ n + e(p), & \text{otherwise.} \end{cases}$$

We have

$$\nu_p(L(ap^{n+1} + b) - L(ap^n + b)) \begin{cases} \geq 2(n + e(p)), & \text{if } b = 0 \text{ and } \rho(p) \mid a, \\ = n + e(p), & \text{if } b = 0 \text{ and } \rho(p) \nmid a, \\ = n + 2e(p) + \nu_p(b), & \text{if } b \neq 0 \text{ and } ap^n + b \equiv 0 \pmod{\rho(p)} \\ & \text{and } n > \nu_p(b), \\ = n + e(p), & \text{if } b \neq 0 \text{ and } ap^n + b \not\equiv 0 \pmod{\rho(p)}. \end{cases}$$

The next theorem describes the cases when the limit does not exist. In preparation for the proof of Theorem 4.17, we prove Lemma 4.15. To complement Theorem 4.13 we need Lemma 4.16 on the least significant p-adic digits of $F(ap^n)$.

Lemma 4.15. *Let $p \neq 2, 5$ be a prime, $\gcd(a,p) = 1$, and $b \neq 0$. The limits $\lim_{n\to\infty} F(ap^n)$, $\lim_{n\to\infty} F(ap^n + b)$ and $\lim_{n\to\infty} L(ap^n + b)$ do not exist if $\left(\frac{5}{p}\right) = -1$, $p \equiv 3 \pmod 4$, and a is an odd multiple of $\rho(p)/2$.*

Lemma 4.16. *For $p \neq 2, 5$, we have*

$$F(ap^{n+1}) \equiv F(ap^n) 5^{\frac{p-1}{2}} \equiv \cdots \equiv F(a) 5^{\frac{p-1}{2}(n+1)} = F(a) \left(\frac{5}{p}\right)^{n+1} \pmod p. \quad (4.8)$$

Therefore, $F(ap^{n+1}) \equiv F(ap^n) \pmod p$, independently of n, exactly if either 5 is a quadratic residue modulo p or $F(a)$ is divisible by p, i.e., $\rho(p) | a$. Otherwise, if $\left(\frac{5}{p}\right) = -1$ then we have $F(ap^{n+1}) \equiv -F(ap^n) \equiv F(ap^{n-1}) \pmod p$, which implies that $F(ap^{n+1}) \not\equiv F(ap^n) \pmod p$ if $\rho(p) \nmid a$.

Now we illustrate some of the differences in the modulo p values of the Fibonacci and Lucas numbers. By (2.2) we get for $a \geq 1$

$$L(ap^n) = \frac{F(2ap^n)}{F(ap^n)} \equiv \frac{F(2a)5^{\frac{p-1}{2}n}}{F(a)5^{\frac{p-1}{2}n}} = \frac{F(2a)}{F(a)} = L(a) \pmod{p}$$

independently of n. We also know that if a is an odd multiple of $\rho(p)/2$ and $\pi(p) \neq 4\pi(p)$ then $L(ap^n)$ is divisible by p; thus, $L(ap^{n+1}) \equiv L(ap^n) \equiv 0 \pmod{p}$; in fact, the congruence holds modulo p^{n+1} by Theorem 1.1 since $\nu_p(L(ap^n)) = n + e(p)$. Of course, by Theorem 4.3 we also know that $\lim_{n\to\infty} L(ap^n)$ exists for any integer $a \geq 1$.

Theorem 4.17. *The limits* $\lim_{n\to\infty} F(a2^n+b)$ *and* $\lim_{n\to\infty} L(a2^n+b)$ *do not exist unless* $a \equiv 0 \pmod{3}$ *when* $\lim_{n\to\infty} F(a2^n+b) = F(b)$*, e.g., it is 0 exactly if* $b = 0$*, and* $\lim_{n\to\infty} L(a2^n+b) = L(b)$*. Otherwise, let* $p \neq 2, 5$ *be a prime,* $\gcd(a,p) = 1$*, and* $\left(\frac{5}{p}\right) = -1$*. For* $b \neq 0$ *the limits* $\lim_{n\to\infty} F(ap^n)$*,* $\lim_{n\to\infty} F(ap^n+b)$*, and* $\lim_{n\to\infty} L(ap^n+b)$ *do not exist if* a *is not a multiple of* $\rho(p)$*. For* $b = 0$*, if* a *is not a multiple of* $\rho(p)$*, then* $\lim_{n\to\infty} F(ap^n)$ *does not exist while* $\lim_{n\to\infty} L(ap^n)$ *does.*

5. Proofs

This section includes the proofs.

Proof of Theorem 2.2. We have that $\pi(2) = 3$ and $\pi(2^n) = 2^{n-1} \cdot 3$ with $n \geq 1$. We have that the difference of indices $(a2^{n+2}+b) - (a2^n+b) = 2^n \cdot 3a$ is a multiple of the period $\pi(2^{n+1})$ and it concludes the proof. □

A similar proof will be presented for Theorem 4.1, too.

Proof of Theorem 2.4. The periodicity of the sequences guarantees the result since $\pi(2^{n+1}) = 2^n \cdot 3$ if $n \geq 0$ and $(a2^n+b) - (a'2^n+b) \equiv 0 \pmod{2^n \cdot 3}$. □

Proof of Lemma 2.7. A proof by induction on $n \geq 0$ proves that

$$L(2^n) \equiv -1 \pmod{2^{n+1}} \tag{5.1}$$

by applying the identity

$$L(2n) + 2(-1)^n = (L(n))^2; \tag{5.2}$$

cf. [21, (17c)]. We note that $\pi(2^{n+1}) = 2^n \cdot 3, n \geq 0$, is a multiple of $2^{n+2} - 2^n = 2^n \cdot 3$; thus, Lemma 2.6, (5.1), and the periodicity yield the results, while (2.1) follows by (5.2), Theorem 1.1, and induction on $n \geq 2$. □

Proof of Lemma 2.9. By identity

$$L(2n) = 2(-1)^n + 5(F(n))^2, n \geq 0, \tag{5.3}$$

(cf. [21, (23)]) we get that $L(2^n \cdot 12a) = L(2(2^{n+1} \cdot 3a)) = 2 + 5(F(2^{n+1} \cdot 3a))^2 \equiv 2 \pmod{2^{2n+6}}$ and $\nu_2(L(2^n \cdot 12a) - 2) = 2n + 6$ by Theorem 1.1. $\square$

Proof of Theorem 2.11. We use (2.3) and (2.4) to get that

$$2F((2^n \cdot 3a + b) + (2^n \cdot 3a)) = F(2^n \cdot 3a + b)L(2^n \cdot 3a) + F(2^n \cdot 3a)L(2^n \cdot 3a + b)$$

and

$$2L((2^n \cdot 3a + b) + (2^n \cdot 3a)) = L(2^n \cdot 3a + b)L(2^n \cdot 3a) + 5F(2^n \cdot 3a)F(2^n \cdot 3a + b).$$

We conclude the proof by dividing by 2 and applying Lemma 2.9, Theorem 1.1, and the periodicity of $L(n)$. We note that in the case of the Lucas numbers the first and the second term on the right hand side provides the 2-adic order in (2.6) if $b = 0$ and $b \neq 0$, respectively. We use $m = a2^n$ and $m' = 2^n \cdot 3a$ in identities (2.3) and (2.4) to derive (2.7) and (2.8), respectively.

The proofs of (2.9) and (2.10) are similar: we use $m = a2^n + b$ and $m' = 2^n \cdot 3a$. By (2.3) and (2.4) we derive that

$$2F(a2^{n+2} + b) = F(a2^n + b)L(2^n \cdot 3a) + F(2^n \cdot 3a)L(a2^n + b)$$

and

$$2L(a2^{n+2} + b) = L(a2^n + b)L(2^n \cdot 3a) + 5F(2^n \cdot 3a)F(a2^n + b)$$

and by dividing by 2, applying Lemma 2.9, and taking the identities modulo 2^{2n+1} we get the desired orders for the rate of convergence.

Note that $(a2^{n+2} + b) - (a2^n + b) \equiv 0 \pmod{12}$ if $n \geq 2$, which implies (2.11). $\square$

Proof of Theorem 2.13. If $a \not\equiv 0 \pmod 3$ then $\lim_{n\to\infty} F(a2^n)$ does not exist by (2.2) of Remark 2.8; thus,

$$\lim_{n\to\infty} F(a2^{2n}) + \lim_{n\to\infty} F(a2^{2n+1}) = 0 \tag{5.4}$$

by Theorem 4.7. However, neither $\lim_{n\to\infty} L(a2^n + b)$ nor $\lim_{n\to\infty}(L(a2^n + b))^2$ exists. In fact neither $\lim_{n\to\infty} L(a2^n + b)$ nor $\lim_{n\to\infty} L(a2^{n+1} + 2b)$ exists by Theorem 4.13 and the latter part proves the nonexistence of the limit $\lim_{n\to\infty}(L(a2^n + b))^2$ by applying identity (5.2).

For the general case, we add up the two addition formulas

$$2F(a2^{2n+1}+b) = F(a2^{2n+1})L(b) + F(b)L(a2^{2n+1})$$

and

$$2F(a2^{2n}+b) = F(a2^{2n})L(b) + F(b)L(a2^{2n}).$$

After taking the limit as $n \to \infty$, we obtain that

$$\begin{aligned}
&2\big(\lim_{n\to\infty} F(a^{2n+1}+b) + \lim_{n\to\infty} F(a2^{2n}+b)\big)\\
&\qquad = L(b)\left(\lim_{n\to\infty} F(a2^{2n+1}) + \lim_{n\to\infty} F(a^{2n})\right) + 2F(b)\lim_{n\to\infty} L(a2^n)\\
&\qquad = -2F(b)
\end{aligned}$$

by (5.4) and Lemma 2.7, which completes the proof. □

Proof of Theorem 2.14. By [15, Corollary 11] and the settings $\phi = (1+\sqrt{5})/2$, $\overline{\phi} = (1-\sqrt{5})/2$, and third roots of unity $\omega(\phi) = -1/2 - \sqrt{-3}/2$ and $\omega(\overline{\phi}) = -1/2 + \sqrt{-3}/2$, we get, after simplifying, the result that

$$\lim_{n\to\infty} F(a2^{2n}+b) = \frac{1}{\sqrt{5}}\left(\omega(\phi)^a\phi^b - \omega(\overline{\phi})^a\overline{\phi}^b\right).$$

□

Proof of Theorem 3.2. Note that $\pi(5^n) = 5^{n-1}\cdot 20$ and $\pi_L(5^n) = 5^{n-1}\cdot 4$, $n \geq 1$; thus, $F(a5^{n+1}+b) \equiv F(a5^n+b) \pmod{5^n}$ and $L(a5^{n+1}+b) \equiv L(a5^n+b) \pmod{5^n}$ by periodicity of the sequences since the difference of the indices is $(a5^{n+1}+b) - (a5^n+b) = 5^n\cdot 4a$, which is a multiple of the periods $\pi(5^n)$ and $\pi_L(5^n)$, respectively. □

Proof of Theorem 3.3. By Theorem 1.1 we immediately get that $\nu_5(F(a5^n)) \geq n$. If $\gcd(a,5) = 1$ then we apply identity [13, (IV.8)]

$$2^{n-1}F_n = \sum_{k=0}^{\frac{n-1}{2}} \binom{n}{2k+1} 5^k, n \geq 1. \tag{5.5}$$

Notice that $\nu_5(\binom{a5^n+b}{2k+1}5^k) \geq n - \lfloor \log_5(2k+1)\rfloor + k$ if $k > (b-1)/2$ and n is sufficiently large. To prove (3.5) we use the fact that $\binom{a5^n+b}{2k+1}5^k \equiv \binom{b}{2k+1}5^k \pmod{5^n}$ for $0 \leq k \leq (b-1)/2$ and the identity (5.5) again. Note that the exponent of 5^n in the last congruence can be increased to $n - \lfloor k/2\rfloor + k$. Also, if $a \equiv 0 \pmod 4$ then the periodicity of $F(n)$ already implies that $\lim_{n\to\infty} F(a5^n+b) = F(b)$. The terms with larger k on the left side are divisible by 5^{n+1} as we have just observed. By (5.5) this leads to the congruence

$$2^{a5^n+b-1}F(a5^n+b) \equiv 2^{b-1}F(b) \pmod{5^{n+1}}$$

and the limit (3.5). This argument can be easily generalized for cases with $b \leq 0$. We observe that $2^{5^n \cdot 2} \equiv -1 \pmod{5^{n+1}}$. Indeed, with any integer $n \geq 0$, we have

$$2^{5^n \cdot 4} - 1 = (2^{5^n \cdot 2} - 1)(2^{5^n \cdot 2} + 1) = ((-1+5)^{5^n} - 1)(2^{5^n \cdot 2} + 1) \equiv 0 \pmod{5^{n+1}}$$

which yields that $2^{5^n \cdot 2} \equiv -1 \pmod{5^{n+1}}$. (The same fact follows by observing that 2 is a primitive root modulo any power of 5 since the Fermat quotient $q_5(2) = (2^4 - 1)/5$ and 5 are relatively prime.) It follows that $\operatorname{inv}(2^{a5^n}, 5^{n+1})$ equals -1 or 1 if $a \equiv 2 \pmod 4$ or $a \equiv 0 \pmod 4$, respectively. In these cases, (3.6) immediately provides the 5-adic limit values $\pm F(b)$. The proof is similar for the Lucas numbers: by the addition formula (2.4)

$$2L(a5^n + b) = L(a5^n)L(b) + 5F(a5^n)F(b)$$

and the second term 5-adically converges to 0. Note that $\operatorname{inv}(2, 5^{n+1}) = 1 + (5^{n+1} - 1)/2$. $\square$

Proof of Lemma 3.4. The proof follows by (5.3) after substituting $5^n \cdot 2a$ and $5^n \cdot a$ into n, respectively, and applying Theorem 1.1. $\square$

Proof of Theorem 3.5. The case with a even is covered by Lemma 3.4. The other part of the statement follows by [14, (IV.9)] since by properly setting the parameters we obtain that $(L(ap^n))^p = L(ap^{n+1}) + \sum_{k=1}^{\frac{p-1}{2}} \binom{p}{k} (-1)^{ap^n k} L(ap^n(p-2k))$ which translates into $(L(a5^n))^5 = L(a5^{n+1}) - \binom{5}{1} L(5^n \cdot 3a) + \binom{5}{2} L(a5^n) = 2^4 L$ for a odd. In fact, by the addition formula (2.4) and Lemma 3.4 we obtain $2L(5^n \cdot 3a) = L(a5^n)L(5^n \cdot 2a) + 5F(a5^n)F(5^n \cdot 2a) \equiv -2L(a5^n) \pmod{5^{2n+1}}$ which yields that $\lim_{n\to\infty} L(5^n \cdot 3a) = -\lim_{n\to\infty} L(a5^n)$ and thus, $L^5 = 2^4 L$. $\square$

Proof of Theorem 3.6. We use (2.3), (2.4), Lemma 3.4, and Theorem 1.1 to derive

$$2F(a5^{n+1} + b) = F(a5^n + b)L(5^n \cdot 4a) + F(5^n \cdot 4a)L(a5^n + b)$$

and

$$2L(a5^{n+1} + b) = L(a5^n + b)L(5^n \cdot 4a) + 5F(5^n \cdot 4a)F(a5^n + b).$$

They imply that

$$2\left(F(a5^{n+1} + b) - F(a5^n + b)\right) = F(a5^n + b)\left(L(5^n \cdot 4a) - 2\right) + F(5^n \cdot 4a)L(a5^n + b)$$

and

$$2\left(L(a5^{n+1} + b) - L(a5^n + b)\right) = L(a5^n + b)\left(L(5^n \cdot 4a) - 2\right) + 5F(5^n \cdot 4a)F(a5^n + b).$$

In order to prove (3.7) and (3.9) we note that $\nu_5(L(a5^n + b)) = 0$ for every $n \in \mathbb{N}$ and $\nu_5(F(a5^n + b)) = 0$ unless $\nu_5(b) > 0$. To prove (3.9) we need a more careful approach since $\nu_5(L(5^n \cdot 4a) - 2) = \nu_5(5F(5^n \cdot 4a)F(a5^n)) = 2n + 1$ by Lemma 3.4

and Theorem 1.1. However, identities (5.2), (2.2), Lemma 3.4, and Theorem 1.1 imply that $2n+1$ is the 5-adic order. In fact, we get that

$$\nu_5(L(a5^{n+1})-L(a5^n)) = \nu_5(5L(a5^n)(F(a5^n))^2\big(3(-1)^{a5^n}+5(F(a5^n))^2\big) = 2n+1.$$

$\square$

Proof of Theorem 4.1. We observe that $\pi(p) \mid (p-1)(p+1)$ since either $\pi(p) \mid p-1$ or $\pi(p) \mid 2(p+1)$; cf. [23, Theorems 6 and 7]. We have that $\pi(p^n) = \pi(p)p^{n-1}$ for any integer $n \geq 1$ if $\pi(p^2) \neq \pi(p)$ (cf. [23] and [3]). It follows that the difference of the indices $(ap^{n+2}+b)-(ap^n+b) = ap^n(p^2-1)$ is a multiple of $\pi(p^{n+1})$ which implies (4.1) and (4.2). $\square$

Proof of Theorem 4.5. The statement follows by [14, (IV.9)] since $(L(n))^3 = L(3n)+3(-1)^nL(n)$ and we substitute $a3^n$ into n. $\square$

Proof of Theorem 4.7. Note that the parity of ap^n+b remains the same for all $n \geq 1$. The statement immediately follows by

$$(L(n))^2 = 4(-1)^n + 5(F(n))^2; \tag{5.6}$$

cf. [21, (24)] or [14, (IV.9)] (note that [13] has a typo in the identity) after substituting $ap^{2n}+b$ and $ap^{2n+1}+b$ into n and taking the limit as $n \to \infty$. If $b \neq 0$ then by Theorem 4.13 we know that $\lim_{n\to\infty} F(ap^{2n}+b) + \lim_{n\to\infty} F(ap^{2n+1}+b) = 0$ happens when $\lim_{n\to\infty} L(ap^n+b)$ does not but $\lim_{n\to\infty}(L(ap^n+b))^2$ does exist.

Of course, both $\lim_{n\to\infty} L(ap^n+b)$ and $L' = \lim_{n\to\infty}(L(ap^n+b))^2$ exist if $b=0$ by Theorem 4.3. $\square$

Proof of Lemma 4.8. We use identity (5.3):

$$L\left(2a\frac{p^2-1}{2}p^n\right) = 2(-1)^{a\frac{p^2-1}{2}p^n} + 5\left(F\big(a\frac{p^2-1}{2}p^n\big)\right)^2$$

and the results follow by Theorem 1.1 since $\rho(p) \mid (p-1)(p+1)/2$. $\square$

Proof of Lemma 4.9. We use identity (5.6):

$$\left(L(a(p-1)p^n)\right)^2 = 4 + 5\left(F(a(p-1)p^n)\right)^2,$$

which implies $\nu_p\left((L(a(p-1)p^n)-2)(L(a(p-1)p^n)+2)\right) = 2(n+e(p))$ by Theorem 1.1, and either $L(a(p-1)p^n) \equiv 2 \pmod{p^{2(n+e(p))}}$ or $L(a(p-1)p^n) \equiv -2 \pmod{p^{2(n+e(p))}}$. The results follow since exactly one of the above factors is divisible by p. A periodicity argument shows that it must be the first or second factor according to $\left(\frac{5}{p}\right) = 1$ or $\left(\frac{5}{p}\right) = -1$, respectively.

In fact, if $\left(\frac{5}{p}\right) = -1$ and $p \equiv 3 \pmod 4$ then $\pi(p) = 2\rho(p)$ and $L(\rho(p)p^n) \equiv -2 = 2\left(\frac{5}{p}\right) \pmod{p^2}, n \geq 0$, by [7, Lemma 3]. We get that $L(a(p-1)p^n) \equiv -2 \pmod{p^{2(n+e(p))}}$ by a periodicity argument which relies on the fact that for the difference of the indices we have $a(p-1)p^n - \rho(p)p^n = \frac{\pi(p)}{2}p^n\left(\frac{p-1}{2}m - 1\right)$, with an odd m, and it is a multiple of the period $\pi(p^{n+1}) = \pi(p)p^n$ (although $\pi(p^2)$ suffices). We could have opted for the use of Lemma 4.11 since now $r = a/(\rho(p)/2)$ is an odd integer, which implies that $L(a(p-1)p^n) \equiv -2 \pmod{p^2}$ by Lemma 4.11.

If $\left(\frac{5}{p}\right) = 1$ then we proceed as above with the periodicity argument changed to $\pi(p) \mid p-1$; therefore, $\pi(p^{n+1}) \mid a(p-1)p^n$ and $L(a(p-1)p^n) \equiv L(0) = 2 = 2\left(\frac{5}{p}\right) \pmod{p^{n+1}}$ without any restriction on $a \geq 1$, e.g., if $\rho(p) \mid a$. (In the latter case (4.3) is also covered by Lemma 4.11.) □

Proof of Lemma 4.11. We use identity (2.4). If m and m' are multiples of $\rho(p)$ then $\nu_p(F(m)F(m')) = \nu_p(mm') + 2e(p)$ by Theorem 1.1. We set $m = r\rho(p)$ and $m' = \rho(p)$; thus,

$$2L((r+1)\rho(p)) \equiv L(r\rho(p))L(\rho(p)) \pmod{p^{2e(p)}}.$$

By induction on $r \geq 1$ we get that $2L((r+1)\rho(p)) \equiv 2\left(\frac{L(\rho(p))}{2}\right)^r L(\pi(p))$ and therefore, $L((r+1)\rho(p)) \equiv 2\left(\frac{L(\rho(p))}{2}\right)^{r+1} \pmod{p^{2e(p)}}$. Note that the exponent is at least 2.

For the remaining part, with $\left(\frac{5}{p}\right) = -1$, we use $L(\rho(p)p^n) \equiv -2 \pmod{p^2}, n \geq 0$, by [7, Lemma 6] if $p \equiv 3 \pmod 4$. We get that $L(r\rho(p)p^n(p-1)/2) \equiv 2\left(\frac{L(\rho(p))}{2}\right)^{rp^n(p-1)/2} = 2(-1)^r \pmod{p^2}$. If $p \equiv 1 \pmod 4$ then $rp^n(p-1)/2$ is already even and we apply (5.2). The case with $\left(\frac{5}{p}\right) = 1$ is implied by the periodicity. □

Proof of Theorem 4.12. By (2.3) we derive that

$$2F(ap^{n+2}+b) = F(ap^n+b)L(a(p^2-1)p^n) + F(a(p^2-1)p^n)L(ap^n+b),$$

and thus,

$$\begin{aligned}&2(F(ap^{n+2}+b) - F(ap^n+b)) \\ &= F(ap^n+b)\left(L(a(p^2-1)p^n) - 2\right) + F(a(p^2-1)p^n)L(ap^n+b).\end{aligned} \tag{5.7}$$

It follows by Lemma 4.8 that $\nu_p(L(a(p^2-1)p^n) - 2)$ is $2(n+e(p))$. Note that $\rho(p) \mid (p-1)(p+1)/2$ which yields that $\nu_p(F(a(p^2-1)p^n)) = n+e(p)$ by Theorem 1.1.

From now on we deal with the p-adic orders of the terms on the right hand side in order to determine the exact p-adic order of the left side.

If $b = 0$ then (5.7) simplifies to

$$2(F(ap^{n+2}) - F(ap^n)) = F(ap^n)\left(L(a(p^2-1)p^n) - 2\right) + F(a(p^2-1)p^n)L(ap^n).$$

We set $h_n = F(ap^{n+2}) - F(ap^n)$. If a is an odd multiple of $\rho(p)/2$ and $\pi(p) \neq 4\rho(p)$, then the p-adic order of both terms is $2(n+e(p))$; thus $\nu_p(h_n) \geq 2(n+e(p))$. Otherwise (i.e., if a is an even multiple of $\rho(p)/2$ or a is not a multiple of $\rho(p)/2$), the p-adic order of the second term and h_n is $n + e(p)$. (Note that $\rho(p)$ is odd exactly if $\pi(p) = 4\rho(p)$.)

If $b \neq 0$ then we have two cases. Now we set $h_n = F(ap^{n+2}+b) - F(ap^n+b)$. If $ap^n + b$ is an odd multiple of $\rho(p)/2$ and $\pi(p) \neq 4\rho(p)$ then the p-adic order of the second term and h_n is $(n+e(p)) + (\nu_p(ap^n+b) + e(p)) = n + 2e(p) + \nu_p(b)$ if $n > \nu_p(b)$, i.e., it is large enough. Otherwise, the p-adic order of the second term and h_n is $n + e(p)$.

For the Lucas numbers we use the addition formula (2.4) and, in a similar fashion to (5.7), we obtain

$$\begin{aligned}2(L(ap^{n+2}+b) - &L(ap^n+b)) \qquad\qquad (5.8)\\ &= L(ap^n+b)\left(L(a(p^2-1)p^n) - 2\right) + 5F(ap^n+b)F(a(p^2-1)p^n).\end{aligned}$$

If $b = 0$ then it simplifies to

$$2(L(ap^{n+2}) - L(ap^n)) = L(ap^n)\left(L(a(p^2-1)p^n) - 2\right) + 5F(ap^n)F(a(p^2-1)p^n).$$

We set $g_n = L(ap^{n+2}) - L(ap^n)$. If a is multiple of $\rho(p)$ then the p-adic order of both terms is $2(n+e(p))$; thus $\nu_p(g_n) \geq 2(n+e(p))$. Otherwise, the p-adic order of the second term and g_n is $n + e(p)$.

If $b \neq 0$ then we have two cases. Now we set $g_n = L(ap^{n+2}+b) - L(ap^n+b)$. If $ap^n + b$ is a multiple of $\rho(p)$ then the p-adic order of the second term and g_n is $(\nu_p(ap^n+b) + e(p)) + (n+e(p)) = n + 2e(p) + \nu_p(b)$ if $n > \nu_p(b)$, i.e., it is large enough. Otherwise, the p-adic order of the second term and g_n is $n + e(p)$. □

Proof of Theorem 4.13. The cases with $p = 2$ and 5 follow by Theorems 2.5 and 3.2. For other primes Theorem 1.1 already implies (4.5) and (4.7). The fact that the limits exist at the same time for any $b \neq 0$ follows by Theorem 4.3 and the addition formulas (2.3) and (2.4):

$$2F(ap^n+b) = F(ap^n)L(b) + F(b)L(ap^n)$$

and
$$2L(ap^n+b) = L(ap^n)L(b) + 5F(ap^n)F(b).$$

If $\left(\frac{5}{p}\right) = 1$ then $\pi(p) \mid p-1$; thus, Remark 4.2 guarantees the existence of the limits for all $a \geq 1$.

If $\rho(p) \mid a$ then we use the above addition formulas. The former one implies that $\lim_{n\to\infty} 2F(ap^n+b) = F(b) \cdot \lim_{n\to\infty} L(ap^n)$ while the latter one shows that $\lim_{n\to\infty} 2L(ap^n+b) = L(b) \cdot \lim_{n\to\infty} L(ap^n)$ and (4.6) follows. □

Proof of Theorem 4.14. We use the addition formulas

$$2F(ap^{n+1}+b) = F(ap^n+b)L(a(p-1)p^n) + F(a(p-1)p^n)L(ap^n+b),$$

and

$$2L(ap^{n+1}+b) = L(ap^n+b)L(a(p-1)p^n) + 5F(ap^n+b)F(a(p-1)p^n),$$

which yield

$$\begin{aligned}2(F(ap^{n+1}+b) - F(ap^n+b)) &= F(ap^n+b)\left(L(a(p-1)p^n) - 2\right)\\ &\quad + F(a(p-1)p^n)L(ap^n+b)\end{aligned}$$

and

$$\begin{aligned}2(L(ap^{n+1}+b) - L(ap^n+b)) &= L(ap^n+b)\left(L(a(p-1)p^n) - 2\right)\\ &\quad + 5F(a(p-1)p^n)F(ap^n+b).\end{aligned}$$

We use Lemma 4.9 and Theorem 1.1 to get the p-adic order on the right hand sides: for the first term and the first factor of the second term, respectively. If either $\left(\frac{5}{p}\right) = 1$ or a is a multiple of $\rho(p)$ then we have $\nu_p(L(a(p-1)p^n - 2) = 2(n+e(p))$ and $\nu_p(F(a(p-1)p^n)) = n + e(p)$. For the other relevant terms we derive that if $b = 0$ then

$$\nu_p(L(ap^n)) = \begin{cases} n+e(p), & \text{if } \left(\frac{5}{p}\right) = 1 \text{ and } a \equiv \rho(p)/2 \pmod{\rho(p)} \text{ and } \pi(p) \neq 4\rho(p),\\ 0, & \text{otherwise.}\end{cases}$$

If $b \neq 0$ then

$$\nu_p(L(ap^n+b)) = \begin{cases} \nu_p(b)+e(p), & \text{if } ap^n+b \equiv \rho(p)/2 \pmod{\rho(p)}\\ & \text{and } \pi(p) \neq 4\rho(p) \text{ and } n > \nu_p(b),\\ 0, & \text{otherwise.}\end{cases}$$

In the application of the second addition identity for the differences of Lucas numbers, we obtain for the last factor that

$$\nu_p(F(ap^n+b)) = \begin{cases} n+e(p), & \text{if } b=0 \text{ and } \rho(p) \mid a, \\ 0, & \text{if } b=0 \text{ and } \rho(p) \nmid a, \\ \nu_p(b)+e(p), & \text{if } b\neq 0 \text{ and } ap^n+b \equiv 0 \pmod{\rho(p)} \\ & \text{and } n > \nu_p(b), \\ 0, & \text{if } b \neq 0 \text{ and } ap^n+b \not\equiv 0 \pmod{\rho(p)}. \end{cases}$$

□

Proof of Lemma 4.15. The addition formula implies

$$2F(ap^{n+1}+b) = F(ap^n+b)L(a(p-1)p^n) + F(a(p-1)p^n)L(ap^n+b). \tag{5.9}$$

If $\left(\frac{5}{p}\right) = -1$, $p \equiv 3 \pmod 4$, and a is an odd multiple of $\rho(p)/2$, then we have $\lim_{n\to\infty} L(a(p-1)p^n) = -2$ by Lemma 4.9 and observe that $\lim_{n\to\infty} F(a(p-1)p^n) = 0$. We assume that $F = \lim_{n\to\infty} F(ap^n+b)$ and $L = \lim_{n\to\infty} L(ap^n+b)$ exist (since Theorem 4.13 states that they exist simultaneously). They cannot be equal to the p-adic 0 by Theorem 1.1 for $b \neq 0$. The limits in (5.9), as $n \to \infty$, lead to a contradiction since $2F \neq -2F$. □

Proof of Lemma 4.16. The congruential identity (4.8) follows by [13, (IV.13)], since

$$F(mp) \equiv F(m)5^{\frac{p-1}{2}} \pmod p$$

for every integer $m \geq 0$. □

Proof of Theorem 4.17. First we assume that $p = 2$ and $b = 0$. It is easy to see that $\lim_{n\to\infty} F(a2^n)$ does not exist unless $a \equiv 0 \pmod 3$ when it is the 2-adic 0. In fact, it follows by the duplication formula (2.2), $F(a2^{n+1}) = F(a2^n)L(a2^n)$. We know that $\lim_{n\to\infty} L(a2^n)$ exists and it is -1 or 2 by Lemma 2.7. In order for $\lim_{n\to\infty} F(a2^n)$ to exist we need that it is either 0 or $\lim_{n\to\infty} L(a2^n) = 1$ which is impossible. However, $F(a2^n)$ is divisible by 2^4 for $n \geq 2$ exactly if $a \equiv 0 \pmod 3$. Theorem 2.5 yields the general case with the extra term b.

If $b \neq 0$, $\left(\frac{5}{p}\right) = -1$ and a is not a multiple of $\rho(p)$ then we set $A = \lim_{n\to\infty} L(a(p-1)p^n)$ and assume that $F = \lim_{n\to\infty} F(ap^n+b)$ exists; thus, $L = \lim_{n\to\infty} L(ap^n+b) \neq 0$ also exists by Theorem 4.13. By Lemma 4.16 we deduce that $\lim_{n\to\infty} F(a(p-1)p^n)$ does not exist: by the assumption that $\rho(p)$ divides neither a nor $a(p-1)$, we get that $F(a(p-1)p^{n+2}) \not\equiv F(a(p-1)p^{n+1}) \pmod p$ and the limit $\lim_{n\to\infty} F(a(p-1)p^n)$ does not exist. However, we get a contradiction since we also have $2F - F \cdot A = L \cdot \lim_{n\to\infty} F(a(p-1)p^n)$ by (5.9).

If $b = 0$, $\left(\frac{5}{p}\right) = -1$ and $\rho(p) \nmid a$ then the proof is similar. We set $A = \lim_{n\to\infty} L(a(p-1)p^n)$, $L = \lim_{n\to\infty} L(ap^n)$, and assume that $F = \lim_{n\to\infty} F(ap^n)$ exists, which will result in a contradiction. The limit F cannot be equal to the p-adic 0 by Theorem 1.1 since a is not a multiple of $\rho(p)$. We take the limit in the addition identity

$$2L(ap^{n+1}) = L(ap^n)L(a(p-1)p^n) + 5F(ap^n)F(a(p-1)p^n)$$

and get that $2L - L \cdot A = 5F \cdot \lim_{n\to\infty} F(a(p-1)p^n)$. Again, we arrive at a contradiction since by Lemma 4.16 we know that $\lim_{n\to\infty} F(a(p-1)p^n)$ does not exist. $\square$

6. More Examples

We include some examples in the Table 1, below, to highlight the use of the theorems and lemmas. We use the following notations

- ND: not defined
- N: no limit
- empty block: the limit exists but not in a simple form (cf. Remark 1.2 and Theorem 2.14)

Acknowledgement. The author thanks the referee for the careful reading of the manuscript, calling his attention to [15] and suggesting the reorganization of this paper which improved its presentation, and Gregory P. Tollisen for his helpful comments.

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p	$\left(\frac{5}{p}\right)$	a	b	$\rho(p)$	$\pi(p)$	$\frac{\pi(p)}{\rho(p)}$	lim $F(ap^n)$	lim $L(ap^n)$	lim $F(ap^n+b)$	lim $L(ap^n+b)$	reason
2	ND	1	0	3	3	1	N	-1	N	-1	T 4.17 L 2.7 T 2.13 T 2.14
2	ND	1	1	3	3	1	N	-1	N	N	T 4.17 L 2.7 T 2.13 T 2.14
2	ND	3	2	3	3	1	0	2	$F(b)(=1)$	$L(b)(=3)$	T 2.5 L 2.7 T 2.14
5	0	1	0	5	20	4	0		0		T 3.3 T 3.5
5	0	1	2	5	20	4	0				T 3.3 T 3.5
5	0	2	75	5	20	4	0	-2	$-F(b)$	$-L(b)$	T 3.3 L 3.4 T 3.5
3	-1	2	0	4	8	2	N	0	N	0	T 4.17 T 4.5 T 4.7 L 4.15
3	-1	2	2	4	8	2	N	0	N	N	T 4.17 T 4.5 T 4.7 L 4.15
3	-1	4	0	4	8	2	0	-2	0	-2	T 4.13 L 4.9 T 4.5
3	-1	4	2	4	8	2	0	-2	-1	-3	T 4.13 L 4.9 T 4.5
3	-1	4	27	4	8	2	0	-2			T 4.13 T 4.5
3	-1	7	1	4	8	2	N		N	N	T 4.17 T 4.5 T 4.7
3	-1	8	1	4	8	2	0	2	1	1	T 4.13 T 4.5 L 4.8
7	-1	4	0	8	16	2	N	0	N	0	T 4.17 L 4.15
11	1	9	121	10	10	1				2	T 4.13
11	1	10	121	10	10	1		2			T 4.13 L 4.9
13	-1	1	0	7	28	4	N		N		T 4.17 T 4.7
13	-1	1	1	7	28	4	N		N	N	T 4.17
13	-1	7	1	7	28	4	0				T 4.13
61	1	1	3	15	60	4					T 4.13
61	1	15	3	15	60	4	0				T 4.13

Table 1: More examples (all the limits are p-adic limits).