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RESEARCH ARTICLES

On Some 2-Adic Properties of a Recurrence Involving Stirling Numbers*

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Abstract—We analyze some 2-adic properties of the sequence defined by the recurrence $Z(1) = 1$; $Z(n) = \sum_{k=1}^{n-1} S(n,k)Z(k)$, $n \geq 2$, which counts the number of ultradissimilarity relations, i.e., ultrametrics on an n-set. We prove the 2-adic growth property $\nu_2(Z(n)) \geq \lceil \log_2 n \rceil - 1$ and present conjectures on the exact values.

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1. INTRODUCTION

The sequence $Z(1) = 1$,

$$Z(n) = \sum_{k=1}^{n-1} S(n,k)Z(k), \quad n \geq 2, \tag{1}$$

was defined as the number of ultradissimilarity relations on an n-set, i.e., the number of not necessarily maximal chains from the minimal to the maximal element in the partition lattice of an n-set in [8] and further discussed in [5, 14], and [16].

Let n be a positive integer, and let $\nu_2(n)$ and $d_2(n)$ denote the highest power of 2 dividing n and the number of ones in the binary representation of n, respectively. Tables 1—3 give the first ten, eighteen, and seven original values, 2-adic orders, and modulo 8 remainders of $Z(n)$, respectively.

n	1	2	3	4	5	6	7	8	9	10
$Z(n)$	1	1	4	32	436	9012	262760	10270696	518277560	32795928016

Table 1. The values of $Z(n)$ for $n \leq 10$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\nu_2(Z(n))$	0	0	2	5	2	2	3	3	3	4	3	3	5	7	9	4	4	4

Table 2. The 2-adic orders of $Z(n)$ for $n \leq 18$.

The asymptotic growth of the sequence $Z(n)$ has been analyzed in [2, 6, 8], and [13]. In this paper, we focus on some particular 2-adic properties of $Z(n)$ that are closely related to those raised in 1998, cf.

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n	1	2	3	4	5	6	7
$Z(n)$ mod 8	1	1	4	0	4	4	0

Table 3. The values of $Z(n)$ mod 8 for $n \leq 7$.

[9]. Our main result is a 2-adic growth property for $Z(n)$ presented in Theorem 7 and Remark 1. More precisely, we show that $\nu_2(Z(n)) \geq \lceil \log_2 n \rceil - 1$ for $n \geq 1$, and $\nu_2(Z(2^n)) \geq n$ for $n \geq 2$. The analysis is based on some 2-adic properties of the Stirling numbers of the second kind whose investigation has attracted significant attention in the last 20 years, e.g., [1, 3, 10, 11], and [12]. At the very heart of the proof of the main result is a lower bound on the 2-adic order of the difference of Stirling numbers determined in [11]. Theorems 9-10 on the parity of particular sequences of Stirling numbers and Theorem 11 on special Stirling number congruences are of independent interest.

We note that expressions that involve weighted sums of the Stirling numbers of the second kind $\{S(n,k)\}_{k=0}^n$ are rarely analyzed from a p-adic point of view due to inherent difficulties. Van Hamme used an umbral calculus based approach to obtain Kummer-type congruential identities for the number of preferential arrangements $a(n) = \sum_{k=0}^n k! S(n,k)$ in [15] but the method does not seem to generalize to our problem. In [4], a conjecture of Wilf that the alternating sum $\sum_{k=0}^n (-1)^k S(n,k)$ is nonzero for all $n > 2$ was discussed. We use a more complex combination of the Stirling numbers in the form of the recurrence (1). The paper concludes by stating some conjectures regarding the structure of the sequence $\nu_2(Z(n))$, $n \geq 1$.

We use the following facts.

Theorem 1 ([3], Theorem 3). *Let $k, n \in \mathbb{N}$ and $1 \leq k \leq n$. Then*

$$\nu_2(S(n,k)) \geq d_2(k) - d_2(n). \tag{2}$$

Theorem 2 ([11], Theorem 5). *Let a, b, and $n \in \mathbb{N}$, $b \leq a$, and n be sufficiently large (in terms of a and b). Then the 2-adic order of $S(a2^n, b2^n)$ becomes constant as $n \to \infty$. In fact, with $g(a,b) = \nu_2(\binom{(2a-b)2^{n-2}-1}{(a-b)2^{n-1}}) = d_2((a-b)2^{n-1}) + d_2(b2^{n-2}-1) - d_2((2a-b)2^{n-2}-1) = d_2(a-b) + d_2(b-1) - d_2(2a-b-1)$, for any $n > \max\{2, g(a,b)+1\}$ we get that*

$$\nu_2(S(a2^n, b2^n)) = g(a,b), \tag{3}$$

and in general,

$$\nu_2(S(a2^n + u, b2^n + u)) = g(a+1, b+1), \tag{4}$$

independently of u, for any integer u: $1 \leq u < 2^n$ as long as $\nu_2(u) > \max\{2, g(a+1,b+1)+1\}$. The periodicity of $g(a,b)$ yields that $\nu_2(S((a+2^t)2^n, b2^n)) = \nu_2(S(a2^n, b2^n))$ if $t \geq \lceil \log_2(2a-b) \rceil$ is a nonnegative integer.

Theorem 3 ([11], Theorem 2). *Let $n, k, c \in \mathbb{N}$ and $1 \leq k \leq 2^n$, then*

$$\nu_2(S(c2^n, k)) = d_2(k) - 1. \tag{5}$$

Theorem 4 ([11], Theorem 11). *Let $n, k \in \mathbb{N}, 3 \leq k \leq 2^n$, u be a nonnegative integer, and $c \geq 1$ be an odd integer, then*

$$\nu_2(S(c2^{n+1} + u, k) - S(c2^n + u, k)) \geq n - \lceil \log_2 k \rceil + 2. \tag{6}$$

Moreover, let $a, b \in \mathbb{N}$ and $a/2 \leq b < a$, then

$$\nu_2(S(a2^n + u, k) - S(b2^n + u, k)) \geq n + \nu_2(a-b) - \lceil \log_2 k \rceil + 2. \tag{7}$$

Theorem 5 ([11], Theorem 13). *For integers $n > m_1 \geq 2$, $m_1 > m_2 \geq 0$, and odd integer $c \geq 1$, we get*

$$\nu_2(S(c2^{n+1}, 2^{m_1} + 2^{m_2}) - S(c2^n, 2^{m_1} + 2^{m_2})) = n - m_1 + 1. \tag{8}$$

Theorem 6 ([11], Theorem 4). *Let $n, c \in \mathbb{N}$ and m be an integer, $2 \le m \le n$, then*

$$S(c2^n, 2^m) \equiv 1 \bmod 4 \tag{9}$$

and for $2 \le m$ with $c2^n > 2^m - 1$,

$$S(c2^n, 2^m - 1) \equiv 3 \cdot 2^{m-1} \bmod 2^{m+1}. \tag{10}$$

2. MAIN RESULTS

Our main result is

Theorem 7. *For $n \ge 2$ and $L \ge 0$ integers, we have $\nu_2(Z(2^n + L)) \ge n$.*

Remark 1. By writing k as $k = 2^n + L$ with $0 \le L \le 2^n$, it follows that $\nu_2(Z(k)) \ge \lceil \log_2 k \rceil - 1$ for $k \ge 1$ integer, and the right hand side bound can be improved by one if $k \ge 4$ is a power of two.

We need some other 2-adic properties of the Stirling numbers of the second kind given below.

Theorem 8. *We have that $S(2^{n+1} + L, 2^n + L)$ is odd for all $n, L \in \mathbb{N}$ with $n \ge 3$ and $0 \le L \le 2^n$.*

Actually, we prove the following generalization.

Theorem 9. *Let $c, n, L \in \mathbb{N}$ with c odd and $n \ge 3$. Then $S(c2^{n+1} + L, c2^n + L)$ is odd for all $0 \le L \le 2^n$ exactly if c is of form $4m + 1$ and m does not have two ones next to each other in its binary representation.*

Proof. We use the recurrence

$$S(N + 1, K + 1) = (K + 1)S(N, K + 1) + S(N, K). \tag{11}$$

In fact, by (11) we have

$$S(c2^{n+1} + L, c2^n + L) = (c2^n + L)S(c2^{n+1} + L - 1, c2^n + L) + S(c2^{n+1} + L - 1, c2^n + L - 1).$$

If $L \ge 2$ is even then the first term on the right hand side is even. Thus, $S(c2^{n+1} + L, c2^n + L)$ and $S(c2^{n+1} + L - 1, c2^n + L - 1)$ share the same parity. On the other hand, if L is odd then by Theorem 1 we have that $\nu_2(S(c2^{n+1} + L - 1, c2^n + L)) \ge \big(d_2(c) + d_2(L)\big) - \big(d_2(c) + d_2(L - 1)\big) \ge 1$ since then $L \le 2^n - 1$, to the same effect as above regarding the shared parity.

In the remaining case of $L = 0$, the condition that c is of form $4m + 1$ and m, written in binary, does not have two ones next to each other guarantees that $S(c2^{n+1}, c2^n)$ is odd for $n \ge 3$ by Theorem 2. In fact, we set $a = 2c$ and $b = c = 2m' + 1$, with some integer $m' \ge 0$, and derive that $\nu_2(S(c2^{n+1}, c2^n)) = g(a, b) = d_2(a - b) + d_2(b - 1) - d_2(2a - b - 1) = d_2(c) + d_2(c - 1) - d_2(3c - 1) = d_2(m') + 1 + d_2(m') - d_2(6m' + 2) = 2d_2(m') + 1 - d_2(3m' + 1)$ by (3). If m' is even then $g(a, b) = 2d_2(m') - d_2(3m')$, and we know that $d_2(m') + d_2(2m') - d_2(3m') \ge 0$ with equality holding exactly if m' and its shifted version $2m'$ avoid matching nonzero digits in their binary forms, proving one part of the claim. In the same vein, after rewriting $g(a, b)$, we get that $g(a, b) = d_2(m') + d_2(2m' + 1) - d_2(3m' + 1)$ is positive if m' is odd since the ones digits of m' and $2m' + 1$ are both equal to 1. Thus, m' must be even. □

Remark 2. The condition $n \ge 3$ cannot be dropped as illustrated by the case with $n = 2$ and $c = 11$ when $\nu_2 S(c2^{n+1} + L, c2^n + L)$ is not constant for $0 \le L \le 2^n$. Clearly, Theorem 8 is a special case of Theorem 9 with $c = 1$.

Numerical experimentation suggests

Conjecture 1. In general, for $c, n, L \in \mathbb{N}$ with c odd and $n \ge 3$, we have that $\nu_2(S(c2^{n+1} + L, c2^n + L))$ is constant for all $0 \le L \le 2^n$.

Remark 3. Under Conjecture 1, the constant value of $\nu_2(S(c2^{n+1}+L, c2^n+L))$ is determined as $2d_2(c) - d_2(3c)$ in identity (3) with $L = 0$ (cf. the proof of Theorem 9) and $L = 2^n$. Identity (4) verifies the same value for $0 < L < 2^n$. Conjecture 1 can be considered as a generalization of Theorem 2. Note that (3) and (4) require lower bounds on $\nu_2(L)$.

We will also need the following

Theorem 10. *With $n \geq 1$ and $L \geq 0$, we have $S(2^{n+1}+L, k)$ is even for all k: $2^n + L < k < 2^{n+1}$. If $L = 0$ we have $\nu_2(S(2^{n+1}, k)) = d_2(k) - 1 \geq 1$ for every $n \geq 1$ and $2^n < k < 2^{n+1}$.*

Proof of Theorem 10. For $L = 0$, we have that $\nu_2(S(2^{n+1}, 2^n + i)) = d(i) \geq 1$, for $1 \leq i \leq 2^n - 1$ by Theorem 3.

Otherwise, assume that we have already proved that $S(2^{n+1}+l, 2^n+l+i)$ is even for every $l \leq L-1$ with some $L \geq 1$ and all i: $1 \leq i \leq 2^n - l - 1$. Now we prove the claim for $l = L$.

We have $S(2^{n+1}+L, 2^n+L+i) = (2^n+L+i)S(2^{n+1}+L-1, 2^n+L+i) + S(2^{n+1}+L-1, 2^n+L+i-1) = (2^n+L+i)S(2^{n+1}+L-1, 2^n+L-1+i+1) + S(2^{n+1}+L-1, 2^n+L-1+i)$ for each i: $1 \leq i \leq 2^n - L - 1$ by identity (11). The second term and the second factor of the first term with $2 \leq i+1 \leq 2^n - L = 2^n - (L-1) - 1$ are even by the assumption. □

Now we are ready to prove the main result.

Proof of Theorem 7. We prove the statement by induction on n. To this end, we assume that $\nu_2(Z(2^n + L)) \geq n$ is true for any integer $L \geq 0$. It follows that $\nu_2(Z(k)) \geq \lceil \log_2 k \rceil - 1$ for all $k \leq 2^n + L$ with $0 \leq L \leq 2^n$, i.e., $k \leq 2^{n+1}$. Now we prove that $\nu_2(Z(2^{n+1}+L)) \geq n+1$ also holds. The problem is that typically, for small values of k, $\nu_2(Z(k))$ is small, and the multiplier $S(2^n+L, k)$ does not seem to help to guarantee high 2-adic orders for the terms in (1). To overcome this problem, we consider the difference

$$\begin{aligned} Z(2^{n+1}+L) - Z(2^n+L) = &\sum_{k=1}^{2^n+L-1} (S(2^{n+1}+L, k) - S(2^n+L, k))Z(k) \\ &+ S(2^{n+1}+L, 2^n+L)Z(2^n+L) \\ &+ \sum_{k=2^n+L+1}^{2^{n+1}+L-1} S(2^{n+1}+L, k)Z(k). \end{aligned} \tag{12}$$

We can 2-adically analyze the terms of the first summand. Indeed, the terms with $k = 1$ and 2 come with large 2-adic orders (cf. Remark 5 in [11]). For a general term with $k \geq 3$, we have

$$\nu_2(S(2^{n+1}+L, k) - S(2^n+L, k)) \geq n - \lceil \log_2 k \rceil + 2$$

by Theorem 4 and $\nu_2(Z(k)) \geq \lceil \log_2 k \rceil - 1$ by the induction hypothesis, which yields the lower bound $n+1$ for the term.

For the first factor of the second term we get that $S(2^{n+1}+L, 2^n+L)$ is odd by Theorem 8.

If $L = 0$ then the second factor of any term in the third summand gives us $\nu_2(Z(k)) \geq \lceil \log_2 k \rceil - 1 \geq n$ by the induction hypothesis and the first factor yields $\nu_2(S(2^{n+1}, k)) = d_2(k) - 1 \geq 1$ by Theorem 10 (or Theorem 3) which guarantees that

$$\nu_2(S(2^{n+1}, k)Z(k)) \geq n+1. \tag{13}$$

By combining the above bounds and applying the induction hypothesis on the 2-adic order of $Z(2^n+L)$, we get that $Z(2^{n+1}) \equiv (1 + S(2^{n+1}, 2^n))Z(2^n) \equiv 0 \bmod 2^{n+1}$.

If $L = 1$ then we split the third summand in (12) into two parts

$$\sum_{k=2^n+L+1}^{2^{n+1}-1} S(2^{n+1}+L, k)Z(k) + \sum_{k=2^{n+1}}^{2^{n+1}+L-1} S(2^{n+1}+L, k)Z(k).$$

The second factor of any term in the first summand gives us $\nu_2(Z(k)) \geq \lceil \log_2 k \rceil - 1 \geq n$ by the induction hypothesis and the first factor yields $\nu_2(S(2^{n+1}+L,k)) \geq 1$ by Theorem 10. For the only term in the second summand, we have just proved that $\nu_2(Z(2^{n+1})) \geq n+1$. By combining the above bounds and applying the induction hypothesis on the 2-adic order of $Z(2^n+L)$, we get that $Z(2^{n+1}+L) \equiv (1+S(2^{n+1}+L,2^n+L))Z(2^n+L) \equiv 0 \bmod 2^{n+1}$ by Theorem 8.

We can proceed for any $L \geq 2$ in a fashion similar to the case with $L=1$. Note that for the terms in the second summand, we have already proved that $\nu_2(Z(2^{n+1}+l)) \geq n+1$ for all $0 \leq l < L$.

We still need to prove the base case. By inspection, we observe that $\nu_2(Z(k)) \geq 2$, for $4 \leq k \leq 7$. In addition, we have that $\nu_2(Z(8+L)) \geq 3$ is true for any integer $L \geq 0$. The proof can be done by induction on L. The case of $L=0$ is verified by Table 2. We observe that $Z(8+L) = \sum_{k=1}^{7+L} S(8+L,k)Z(k)$. By the induction hypothesis, it is sufficient to prove that $\nu_2(\sum_{k=1}^{7} S(8+L,k)Z(k)) \geq 3$, more precisely, by Table 3, that $S(8+L,1)+S(8+L,2)+4\,(S(8+L,3)+S(8+L,5)+S(8+L,6)) \equiv 0 \bmod 8$ which can be easily verified by using the equation

$$S(N,K) = \frac{1}{K!}\sum_{i=0}^{K}\binom{K}{i}(-1)^i(K-i)^N.$$

□

3. OPEN PROBLEMS

We suggest three conjectures that are still open, cf. [9].

Conjecture 2. For $n \geq 3$, we have $\nu_2(Z(2^n)) = n$.

Conjecture 3. For $n \geq 2$, we have that

$$3 \cdot 2^{n-1} \leq \max\{k \mid \nu_2(Z(k)) = n\} \leq 2^{n+1}-1.$$

We believe that the following stronger version, which claims that the actual value of $\max\{k \mid \nu_2(Z(k)) = n\}$ is the lower bound, holds true.

Conjecture 4. For $n \geq 2$, we have that

$$\max\{k \mid \nu_2(Z(k)) = n\} = 3 \cdot 2^{n-1}.$$

Remark 4. Conjecture 4 slightly improves the lower bound on $\nu_2(Z(k))$ given in Remark 1, and we get that $\nu_2(Z(k)) \geq \lceil \log_2 k/3 \rceil + 1$ for $k \geq 3$.

Remark 5. If $\nu_2(S(2^{n+1},k)Z(k)) = n+1$ for some k: $2^n < k < 2^{n+1}$ with $n \geq 2$, then in terms of $\nu_2(Z(k))$, we have $\nu_2(Z(k)) = n - d_2(k) + 2$ by Theorem 3. On the other hand, $\nu_2(Z(k)) \geq n$ holds true in this range by Remark 1, and thus, $d_2(k) \leq 2$ and $k = 2^n + 2^a$ with some a: $0 \leq a \leq n-1$.

Now, we extend Conjectures 2 and 4. The following conjecture was discovered by numerical experimentation (cf. Table 4 for small values).

Conjecture 5. If $\nu_2(S(2^{n+1},k)Z(k)) = n+1$ for some k: $2^n < k < 2^{n+1}$ with $n \geq 2$, then in Remark 5, we have $a \neq n-1-a$. In this case $\nu_2(S(2^{n+1},2^n+2^{n-1-a})Z(2^n+2^{n-1-a})) = n+1$ also holds. In the special case of $a=0$, it appears that we always have $\nu_2(S(2^{n+1},2^n+1)Z(2^n+1)) = \nu_2(S(2^{n+1},3 \cdot 2^{n-1})Z(3 \cdot 2^{n-1})) = n+1$, i.e., $\nu_2(Z(2^n+1)) = \nu_2(Z(3 \cdot 2^{n-1})) = n$.

Remark 6. Note that $\nu_2(S(2^{n+1},k)Z(k)) = \nu_2(S(2^{n+N},k)Z(k))$ for any integer $N \geq 1$ by Theorem 3, e.g., if $\nu_2(S(2^{n+1},k)Z(k)) = n+1$ then $\nu_2(S(2^{n+N},k)Z(k)) = n+1$, too.

Remark 7. Conjecture 5 can be equivalently expressed in terms of $\nu_2(Z(k))$: if $\nu_2(Z(k)) = m - d_2(k) + 2$ with some k: $2^m < k < 2^{+1}$ then in Remark 5, we have $a \neq m-1-a$. In this case $\nu_2(Z(2^m + 2^{m-1-a})) = m - d_2(k) + 2$ also holds. For such k, of the form $k = 2^m + 2^a$ and $k = 2^m + 2^{m-1-a}$, we have that $\nu_2\left((S(2^{n+1},k) - S(2^n,k))Z(k)\right) = (n-(m+1)+2) + (m-2+2) = n+1$ for any $n \geq m+1$ by Theorem 5. It seems that the special case with $a = 0$ always yields $\nu_2(Z(2^m+1)) = \nu_2(Z(3 \cdot 2^{m-1})) = m$, however, deriving this appears to be harder than proving Conjecture 2.

We also need the simplified version of Conjecture 2 and (17) from [11].

Conjecture 6 ([11]). Let $n, k, a, b \in \mathbb{N}$, $n \geq 3$, and $3 \leq k \leq 2^n$, then

$$\nu_2(S(2^{n+1},k) - S(2^n,k)) = n + 1 - f(k) \tag{14}$$

for some function $f(k)$ which is independent of n:

$$f(k) = 1 + \lceil \log_2 k \rceil - d_2(k) - \delta(k), \tag{15}$$

with $\delta(4) = 2$ and otherwise it is zero except if k is a power of two or one less, in which cases $\delta(k) = 1$.

Note that $\lceil \log_2 k \rceil - d_2(k)$ is the number of zeros in the binary expansion of k, unless k is a power of two. The function $f(k)$ has been determined in [11] for small values of k: $f(3) = f(4) = 0$, and $f(5) = f(6) = 2$.

We note that Conjecture 5 extends Conjecture 4.

Proof of Conjecture 3 under Conjectures 5-6. According to Conjecture 5 and Remark 7, $\nu_2(Z(3 \cdot 2^{n-1})) = n$ holds for $n \geq 2$. By Remark 1, we have that $\nu_2(Z(2^{n+1})) = n+1$ for $k = 2^{n+1}$ and $\nu_2(Z(k)) \geq \lceil \log_2 k \rceil - 1 \geq n+1$ for any $k > 2^{n+1}$. □

Now we present the proof of Conjecture 2 under Conjectures 5–6.

Proof of Conjecture 2 under Conjectures 5–6. We prove by induction on $n \geq 3$. We assume that $\nu_2(Z(2^N)) = N$ for any $N, 3 \leq N \leq n$. Table 2 verifies that $\nu_2(Z(8)) = 3$ indeed. For $n \geq 3$, we use the summation (12) with $L = 0$:

$$\begin{aligned} Z(2^{n+1}) - Z(2^n) = &\sum_{k=1}^{2^n-1} (S(2^{n+1},k) - S(2^n,k))Z(k) \\ &+ S(2^{n+1},2^n)Z(2^n) \\ &+ \sum_{k=2^n+1}^{2^{n+1}-1} S(2^{n+1},k)Z(k). \end{aligned} \tag{16}$$

For the third summand, we revisit the case of $L = 0$ in the proof of Theorem 7. Now, for a general term, we improve the lower bound (13). In fact, due to Conjecture 5, we have the refined 2-adic lower bound $\nu_2(S(2^{n+1},k)Z(k)) \geq n+2$ except possibly if $k = 2^n + 2^a$ and then $k = 2^n + 2^{n-1-a}$ too, in which case by appropriately pairing these exceptional terms, we get
$\nu_2\left(S(2^{n+1}, 2^n + 2^a)Z(2^n + 2^a) + S(2^{n+1}, 2^n + 2^{n-1-a})Z(2^n + 2^{n-1-a})\right) \geq n+2$ again.

In the first summand, the terms with $k = 1$ and 2 have large 2-adic orders according to Remark 5 of [11]. For $k \geq 3$, we use the lower bound $\nu_2(Z(k)) \geq \lceil \log_2 k/3 \rceil + 1$ given in Remark 4, and get $\nu_2\left((S(2^{n+1},k) - S(2^n,k))Z(k)\right) \geq (n - \lceil \log_2 k \rceil + 2) + (\lceil \log_2 k/3 \rceil + 1) = n+2$ for every k except if k: $2^m < k \leq 3 \cdot 2^{m-1}$ for some $m : 1 \leq m \leq n-1$. Note that in the given range $d_2(k) \geq 2$.

Now if $\nu_2\left((S(2^{n+1},k) - S(2^n,k))Z(k)\right) = n+1$ then since $\nu_2(S(2^{n+1},k) - S(2^n,k)) \geq n - \lceil \log_2 k \rceil + 2$ by Theorem 4, it follows that $\nu_2(Z(k)) \leq \lceil \log_2 k \rceil - 1$. According to Remark 1, then we must have

$\nu_2(Z(k)) = \lceil \log_2 k \rceil - 1 = m$; thus, $\nu_2(S(2^{n+1}, k) - S(2^n, k)) = n - m + 1$. This yields $d_2(k) = 2$ by Conjecture 6.

Therefore, the condition $\nu_2(Z(k)) = m - d_2(k) + 2 = m$ in Remark 7 is satisfied. Now $k = 2^m + 2^a$ and its pair $k = 2^m + 2^{m-1-a}$ both yield $\nu_2(Z(2^m + 2^m)) = \nu_2(Z(2^m + 2^{m-1-a})) = m$. Theorem 5 guarantees that $\nu_2\left(S(2^{n+1}, 2^m + 2^a) - S(2^n, 2^m + 2^a)\right) = \nu_2(S(2^{n+1}, 2^m + 2^{m-1-a}) - S(2^n, 2^m + 2^{m-1-a})) = n - m + 1$, and the combined term

$$\begin{gathered}\left(S(2^{n+1}, 2^m + 2^a) - S(2^n, 2^m + 2^a)\right) Z(2^m + 2^a) + \\ \left(S(2^{n+1}, 2^m + 2^{m-1-a}) - S(2^n, 2^m + 2^{m-1-a})\right) Z(2^n + 2^{m-1-a})\end{gathered}$$

has a 2-adic order of at least $n + 2$.

Finally, for $n \geq 3$, we have $S(2^{n+1}, 2^n) \equiv 1 \bmod 4$ in the second summand of (16) by Theorem 6. Thus, we conclude that

$$Z(2^{n+1}) \equiv \left(1 + S(2^{n+1}, 2^n)\right) Z(2^n) \equiv 2Z(2^n) \mod 2^{n+2} \tag{17}$$

holds by (16) which completes the proof by the induction hypothesis. □

Since $\nu_2(1 + S(2^{n+1} + 1, 2^n + 1)) \geq 2$ for $n \geq 4$ by identity (11), the summation (12) fails to provide the proof for $\nu_2(Z(2^n + 1))$, i.e., if $L = 1$ in Conjecture 5. We note that the following improvement of Theorem 6 might help in further generalizations of (17).

Theorem 11. *With $n \in \mathbb{N}$ and $n \geq 4$, we have*

$$1 + S(2^{n+1}, 2^n) \equiv 6 \bmod 32.$$

The proof is based on identity (9) of [12] with the setting $m = n - 1 \geq 3$ and $a = 1$, which yields that

$$S(2^n, 2^{n-1}) \equiv \binom{3 \cdot 2^{n-3} - 1}{2^{n-2}} \bmod 2^{n-1}.$$

Then, it concludes by using the generalization of the classical theorem on $\binom{N}{M}/p^{\nu_p\left(\binom{N}{M}\right)} \bmod p$ to modulo any prime power p^q, with $p = 2$ and $q = 5$, cf. [7, Theorem 1].

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$\nu_2(Z(n))$	0	0	2	5	2	2	3	3	3	4	3	3	5	7	9	4	4
$\nu_2(S(2^6, n)Z(n))$	0	0	3	5	3	3	5	3	4	5	5	4	7	9	12	4	5
$\lceil \log_2 n \rceil$	0	1	2	2	3	3	3	3	4	4	4	4	4	4	4	4	5

n	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34
$\nu_2(Z(n))$	4	4	4	4	6	4	4	7	5	5	8	6	8	6	5	5	7
$\nu_2(S(2^6, n)Z(n))$	5	6	5	6	8	7	5	9	7	8	10	9	11	10	5	6	8
$\lceil \log_2 n \rceil$	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	6	6

Table 4. The values of $\nu_2(S(2^6, k)Z(k))$ for $k \leq 34$.

We note that for $p \neq 2$, $\nu_p(Z(n))$ does not seem to have any transparent structure and behaves quite chaotically. Finally, we mention the (divergent) exponential generating function of the sequence $Z(n), n \geq 1$,

$$\mathcal{Z}(x) = \sum_{n=1}^{\infty} Z(n)\frac{x^n}{n!}$$

that satisfies the functional equation $2\mathcal{Z}(x) = x + \mathcal{Z}(e^x - 1)$ as noted in [8]. It can be used to analyze the growth of the sequence $Z(n)$, cf. [6] and [13]. It would be interesting to see if it can also help in discovering p-adic properties of the sequence.

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