Direct consequences of the basic ballot theorem

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Abstract

We use only the classic basic ballot result and simple combinatorial arguments to derive the distributions of the first passage time and the number of visits in the usual random walk model.

1. Introduction

In the usual simple ballot problem (cf. Feller (1968)), there are two candidates A and B, and $a+b$ voters, casting their votes one after the other in some order. Candidate A gets a votes while B collects b votes during one of the possible $\binom{a+b}{a}$ ordered voting sequences with the given voting outcome. In an actual instance of the voting process, we count the respective numbers of votes α_r for A and β_r for B among the first r votes, $r = 1, 2, \ldots, a+b$ (cf. Saran and Sen (1985)). Furthermore, for $-b \le c \le a$, let $\delta_{a,b}^{(c)}$ denote the number of subscripts $r = 1, 2, \ldots, a+b$ for which $\alpha_r = \beta_r + c$. For example, $\delta_{a,b}^{(0)}$ counts the number of times the vote counts coincide (after the first vote has been cast).

Theorem 1 (Ballot Theorem). *Assuming that all $\binom{a+b}{a}$ orderings of the votes are equally likely and $a \ge b$, we have*

$$P(\delta_{a,b}^{(0)} = 0) = P(\alpha_r > \beta_r, r = 1, 2, \ldots, a+b) = \frac{a-b}{a+b}.$$

There are many generalizations of the above theorem involving linear functions of α_r and β_r in the definition of the underlying events. Here we focus on certain generalizations of Theorem 1 that can be proven by this theorem combined with simple combinatorial arguments.

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Voting sequences can be described by random walks. There are two usual ways of representing a random walk. In both models, the random walk starts at $(0, 0)$ and makes a and b moves of the first and second types, respectively. The order of the moves of the two types is the same in both models; however, the steps are represented differently.

In the first model, a legal move is either a right move $(1, 0)$ or an upward move $(0, 1)$ of unit length. If the random walk makes a and b moves of the respective types then the random walk ends at the point (a, b).

In the second model, in each step the random walk makes a SE-move $(1, -1)$ or a NE-move $(1, 1)$ (of length $\sqrt{2}$). If the random walk makes a and b moves of the respective types then it now ends at the point $(a + b, b - a)$. In the first model horizontal steps represent the votes for A while in the second model SE steps correspond to these votes.

We note that a simple probabilistic proof of Theorem 1 follows immediately without applying any direct combinatorial path enumeration. We use the first model and observe that any bad path (i.e., a path with $\delta_{a,b}^{(0)} > 0$) either starts with a vote for B, or starts with a vote for A and will meet the line $y = x$ at some point in the voting process. (If one uses the second model then this line is replaced by the x-axis.) By reflecting the first such segment with respect to this line, when beginning with a vote for B, we face another bad path starting with a vote for A, and vice versa. Thus, the probability of a bad path is equal to twice the probability $b/(a+b)$ that the voting sequence starts with a vote for B. We immediately get that $P(\delta_{a,b}^{(0)} = 0) = 1 - 2b/(a + b) = (a - b)/(a + b)$. We remark that proofs of the generalizations of the basic Ballot Theorem that rely on combinatorial path enumeration often use a similar but more involved uniform partitioning of the set of bad paths.

We use only the Ballot Theorem and simple combinatorial arguments to derive the distributions of the first passage time and the number of visits in Sections 2 and 3, respectively, in the usual random walk model.

2. First passage

The second model is visually more helpful when the focus is on the difference between the numbers of votes. Let $X_i = 1$ or -1 be corresponding to a NE or a SE move, respectively, in the second model. We define the successive partial sums $S_0 = 0, S_1, S_2, \ldots, S_{a+b}$, with $S_k = \sum_{i=1}^{k} X_i, k = 0, 1, \ldots, a + b$, which represent the successive cumulative gains from the perspective of candidate B, i.e., $S_k = \beta_k - \alpha_k$. It can be used to analyze the first passage time distribution. We say that a first passage through m occurs at step k if for the partial sums we have

$$S_1 < m, S_2 < m, \ldots, S_{k-1} < m, S_k = m$$

for $m > 0$, and it is defined similarly for $m < 0$ with all inequalities switched.

Clearly, with appropriately scaling the models (cf. Figure 1), both models can be used for enumerating random walks of certain types.

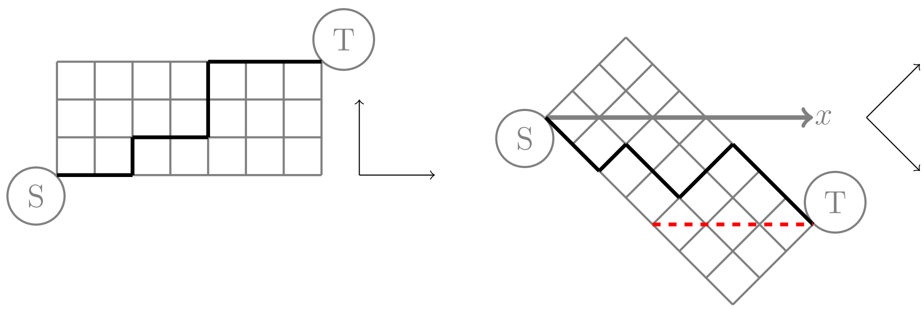

Figure 1: Two representations of the same ballot sequence of 10 votes with $a = 7$ and $b = 3$ by random walks starting at (0,0) and ending at (7,3) and (10,-4), respectively, in the two models with the directions of the legal moves illustrated on the right of the grids. In the first model, we immediately see that $\delta^{(0)}_{7,3} = 0$. In the second model, we also notice that the vote counts have not coincided; moreover, we can see that the random walk achieves first passage through -4 at its very end. We will use this correspondence since the first model offers the applicability of the ballot theorem.

In Figure 2 below we consider the cases $(a, b) = (7, 3)$ and (3,7), respectively, in the first model. Here S and T stand for start and terminal nodes, and the walk is supposed to stay within the gray area except for its first (right) step or last (upward) step, depending on whether $a > b$ or $a < b$ where a and b indicate the number of horizontal $(1, 0)$ and vertical $(0, 1)$ moves of unit length, respectively. We draw a random walk with the (dashed red) forbidden line (i.e., a line that the random walk is not supposed to touch except at the beginning or at the end provided that $a > b$ or $a < b$, respectively) emphasized and dashed. Obviously, if we reverse a random walk by changing the role of T and S and take the steps in reverse order, which also corresponds to a proper reflection (cf. Figure 3) about the line $y = -x$, then the number of random walks with the above stipulation (i.e., staying within the gray area except for the first or last step) are the same. Note that a different kind of reversal leads to the notion of a dual walk (Feller, 1968, pp.91-92).

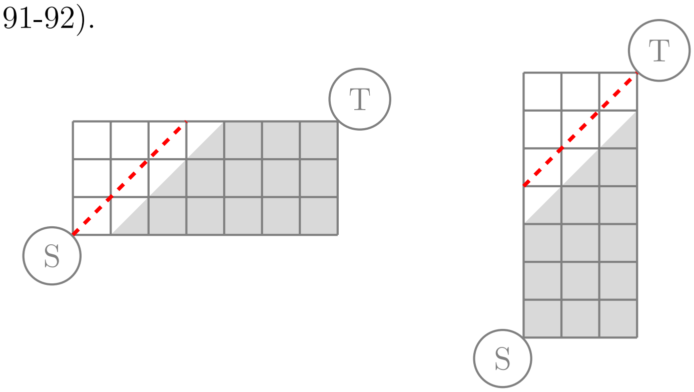

Figure 2: Ballot problems with 7 vs. 3 votes with candidate A always leading, starting from the first vote, and 3 vs. 7 votes with candidate B leading by the largest margin only at the end

We can address first passage problems by reducing them to ballot problems. In fact, based on the right panel representation of the random walk in Figure 1 and its equivalent image in the right panel in Figure 2 reflected about the y axis and then rotated by 135° in the clockwise direction,

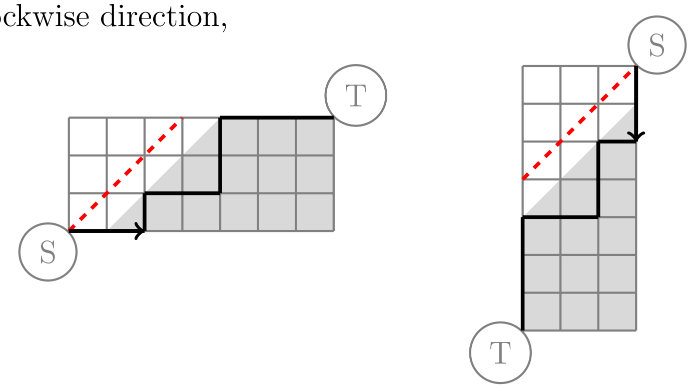

Figure 3: A random walk for the ballot problem with 7 vs. 3 votes (with A always in the lead), and the same walk differently drawn after proper reflection

we can count

$$\#\{\text{walks with first passage through } m \geq 1 \text{ at step } 2n+m\} = \frac{m}{2n+m}\binom{2n+m}{n} \quad (1)$$

where the last factor $\binom{2n+m}{n}$ counts the number $N_{2n+m,m}$ of random walks from the origin to the point $(2n+m, m)$ in the second model. Indeed, we set $m = b - a$ and

$2n+m = a+b$, i.e., $a = n$ and $b = m+n$, and use the Ballot Theorem (i.e., Theorem 1) which yields $(b-a)/(a+b)\binom{a+b}{a} = m/(2n+m)\binom{2n+m}{n}$ for the number of random walks with a and b votes for the respective candidates and reaching the largest difference at the end. In the so-called symmetric random walk model in which both moves are equally likely at each step, this yields

Theorem 2.

$$P(\textit{first passage through } m \geq 1 \textit{ occurs at step } 2n+m) = \frac{m}{2n+m}\binom{2n+m}{n}2^{-(2n+m)}.$$

Clearly, one can easily obtain the corresponding result for any random walk with moves $X_i = \pm 1$, even in the asymmetric case. Typically, most sources including (Feller, 1968, the proof of Theorem 2, p.89) derive Theorem 2 by using "a trite calculation" based on $N_{2n+m,m-1}$, $N_{2n+m,m+1}$, and related probabilities (cf. (Feller, 1968, identities (7.4) and (7.5) on p.89)) while this proof is a straightforward application of the Ballot Theorem.

We note that Addario-Berry and Reed (2008) reviews the connections between ballot-style problems and random walks conditioned on the value of S_n and also suggests taking the votes in reverse order for the proof of Theorem 2. Other proving techniques are presented, e.g., the one based on the cyclic arrangements of the steps X_is due to Dvoretzky and Motzkin (1947) that is a powerful tool in proving various generalizations. Other forms of the Ballot Theorem are given, e.g., the hitting time theorem for left-continuous random walks due to Kemperman (1961) which includes classical games of chance with betting a single unit in every play.

Theorem 3 (Hitting Time Theorem, Kemperman (1961))**.** *For a random walk starting in $k \geq 1$ with i.i.d. steps $\{X_i\}_{i=1}^{\infty}$ satisfying $X_i \geq -1$, the conditional probability that the walk hits the origin for the first time at time n, given that it does hit zero at time n, is equal to k/n. Equivalently, the chances that the random walk visits $-k$ for the first time at n is k/n of the chance that without qualification it visits $-k$ at n.*

For an elementary proof and background see van der Hofstad and Keane (2008). We note that Theorem 3 immediately follows for random walks with moves $X_i = \pm 1$ only by representation as a first passage problem. In fact, by counting walks based on (1) we get that the probability of first passage to $-k$ at step n given passage to $-k$ at step n is

$$\frac{k}{n}\binom{n}{\frac{n+k}{2}}\Big/\binom{n}{\frac{n+k}{2}} = \frac{k}{n},$$

since there are $(n+k)/2$ and $(n-k)/2$ moves of types $X_i = -1$ and $X_i = 1$, respectively.

3. Number of visits

Now we turn to questions regarding the number of visits, i.e., the number of times a random walk reaches a certain height in the second model; or in other words, how frequently the difference in the number of votes $S_k = \beta_k - \alpha_k, k = 1, 2, \dots, a + b$, equals a certain value is counted. We can use the above ballot correspondence to prove more involved distributional problems and determine all probabilities $P(\delta_{a,b}^{(c)} = j)$. We pick some relations from Saran and Sen (1985), more precisely, identities (23) of case (ii), (16), and Theorem 9, and apply the approach from the previous section. Note that in Saran and Sen (1985) more involved calculations are used, and the authors' main purpose was to derive results for generalized ballot problems rather than applying the basic Ballot Theorem to prove them. We derive the distribution of the number of visits to c in

Theorem 4. *If $c > 0$ then we have that*

$$P(\delta_{a,b}^{(0)} = j) = 2^j \frac{a-(b-j)}{a+(b-j)} \binom{a+b-j}{a} \Big/ \binom{a+b}{a} = 2^j \frac{a-b+j}{a+b-j} \binom{a+b-j}{a} \Big/ \binom{a+b}{a}, \text{ for } a \geq b \text{ and } 0 \leq j \leq b; \tag{2}$$

$$P(\delta_{a,b}^{(c)} = j) = 2^{j-1} \frac{a-(b-(j-1))}{a+(b-(j-1))} \binom{a+b-(j-1)}{a} \Big/ \binom{a+b}{a} = 2^{j-1} \frac{a-b+j-1}{a+b-j+1} \binom{a+b-j+1}{a} \Big/ \binom{a+b}{a}, \tag{3}$$

$$\text{for } a \geq b + c, \text{ and } 1 \leq j \leq b + 1;$$

and

$$\begin{aligned} P(\delta_{a,b}^{(-c)} = j) &= 2^{j-1} \frac{a+c-(b-c-(j-1))}{a+c+(b-c-(j-1))} \binom{a+c+b-c-(j-1)}{a+c} \Big/ \binom{a+b}{a}, \\ &= 2^{j-1} \frac{a+2c-b+j-1}{a+b-j+1} \binom{a+b-j+1}{a+c} \Big/ \binom{a+b}{a}, \end{aligned} \tag{4}$$

$$\text{for } a \geq b - c \geq 0 \text{ and } 1 \leq j \leq b - c + 1.$$

Note that identity (2) yields Theorem 1 with $j = 0$ as a special case, and it gives the distribution of all paths according to the number of meeting points with the line $y = x$ excluding the starting point S. (By convention, we always exclude the starting point when counting the meeting points.) A lovely application of this distribution with $a = b$ can be found in Zagier (1990) in which the distribution is derived by using its generating function.

We prove identities (2) and (3), and show that (4) can be reduced to an application of (3). In addition, the proof of Theorem 5 illustrates that identity (3) can be further reduced to (2).

Note that the distribution of $\delta_{a,b}^{(c)}$ in the missing case with $a-b<c\leq a$ can be easily derived by using the notion of the dual path (cf. (Feller, 1968, pp.89-90)) and (4). (Sometimes, taking the dual is referred to as a rotation.) For example, we obtain that

$$P(\delta_{a,b}^{(c)}=a-c+1)=P(\delta_{a,b}^{(a-b-c)}=a-c+1)=\frac{2^{a-c}}{\binom{a+b}{a}}.$$

If $a=b$ then we cannot apply formula (3) and end up using this approach based on (4).

Proof of Theorem 4.

Proof of identity (2). We may assume that $j\geq 1$. We borrow the notion of the representative path from (Feller, 1968, III.7 Maxima and first passages, p.90) and adapt its proof of the fact that the probability of the rth return to the origin occurs at epoch n is given by the probability that the first passage through r occurs at epoch $n-r$ (cf. (Feller, 1968, Theorems 2 and 4, pp.89-90)). A path from $S=(0,0)$ to $T=(a,b)$ with $\delta_{a,b}^{(0)}=j$ is said to be representative if all segments are below the line $y=x$, and there are exactly j meetings with this line. In this case, the representative path consists of j sections with endpoints on $y=x$, and 2^j different paths can be constructed by mirroring the sections on this line and still satisfying $\delta_{a,b}^{(0)}=j$.

From the representative path, we remove the j moves that end on this line; thus, we obtain a new path with $\delta_{a,b-j}^{(0)}=0$. The procedure can be reversed by inserting j upward moves, one at each point where the path leaves the $y=x-i, i=1,2,\ldots,j$, lines for the last time. Clearly, for the number of representative paths after the removals we get $\binom{a+b-j}{a}P(\delta_{a,b-j}^{(0)}=0)=(a-b+j)/(a+b-j)\binom{a+b-j}{a}$ by Theorem 1, and the proof is complete.

Proof of identity (3). Note that the probability in (3) is independent of c; thus, it suffices to prove it for $c=1$ and show the independence for all integers $c: 0<c\leq a-b$.

The proof of identity (2) can be easily revised. We add an extra vote for B. Now we consider all paths P' going from (0,0) to $(a,b+1)$ with $\delta_{a,b+1}^{(0)}=j$. If the first vote cast is for B then the actual path can be viewed as a path P going from (0,0) to (a,b) with $\delta_{a,b}^{(1)}=j$. This operation can be reversed if one wants to construct a path P' from P. Recalling the notion of representative paths for the new problem and enumerating the paths P', we have a 2^{j-1} factor rather than 2^j since the first vote should be for B. This means that the first segment of P', until meeting the line $y = x$, has B in the lead. This operation yields, via identity (2), that the number of paths with $\delta^{(1)}_{a,b} = j$ is

$$\binom{a+b}{a} P(\delta^{(1)}_{a,b} = j) = \frac{1}{2}\binom{a+b+1}{a} P(\delta^{(0)}_{a,b+1} = j) = 2^{j-1}\frac{a-b-1+j}{a+b+1-j}\binom{a+b+1-j}{a}.$$

Now we are ready to prove the statement for any $c : 2 \le c \le a-b$. We can define a one-to-one correspondence between paths with $\delta^{(c)}_{a,b} = j$ and $\delta^{(c+1)}_{a,b} = j$, $1 \le c \le a-b-1$: take any path P with $\delta^{(c)}_{a,b} = j$ and identify the point where P meets the line $y = x - c$ for the first time. We add an extra vote for A to P at this point and remove the last vote for A in P: we get a path with $\delta^{(c+1)}_{a,b} = j$. Clearly, this operation can be reversed.

Proof of identity (4). We present a one-to-one correspondence between the paths $\delta^{(-c)}_{a,b} = j$ and $\delta^{(c)}_{a+c,b-c} = j$. To this end, with respect to the line $y = x + c$, we take the mirror image of the leading segment, ending at the last meeting point with this line, of all paths with $\delta^{(-c)}_{a,b} = j$. The reflections yield another grid with side lengths $a' = a + c$ and $b' = b - c$, and paths with $\delta^{(c)}_{a',b'} = j$. We illustrate this step on Figure 4, with $a = 7, b = 3, c = 1$, and $j = 2$.

By the above correspondence and identity (3), we have that

$$\begin{aligned}\tbinom{a+b}{a} P(\delta^{(-c)}_{a,b} = j) &= \tbinom{a'+b'}{a'} P(\delta^{(c)}_{a'=a+c,b'=b-c} = j) = 2^{j-1}\tfrac{a'-b'+j-1}{a'+b'-j+1}\tbinom{a'+b'-j+1}{a'} \\ &= 2^{j-1}\tfrac{a-b+2c+j-1}{a+b-j+1}\tbinom{a+b-j+1}{a+c}.\end{aligned}$$

□

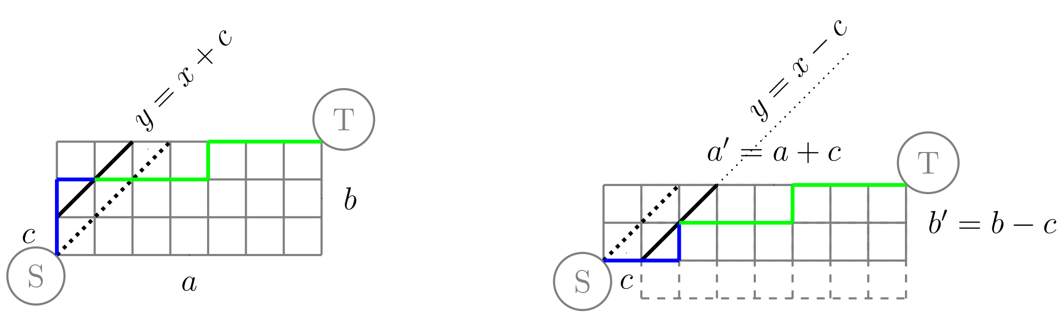

Figure 4: For $(a, b) = (7, 3)$, visits to $y = x + 1$ vs. visits to $y = x - 1$ with $c = 1$, $a' = a + c$ and $b' = b - c$

It is worth noting the surprising connection between identities (2) and (3) that we state as a theorem of independent interest and present a short proof based on taking the partial dual of the corresponding paths.

Theorem 5. *We have for* $0 < c \leq a - b$ *and* $1 \leq j \leq b + 1$ *that*

$$P(\delta_{a,b}^{(c)} = j) = P(\delta_{a,b}^{(0)} = j - 1).$$

Proof of Theorem 5. We consider a path P that visits the line $y = x - c$ for the last time at the point $(d+c, d)$ for some integer $d > 0$. We define a one-to-one correspondence between the paths with $\delta_{a,b}^{(c)} = j$ and $\delta_{a,b}^{(0)} = j - 1$. To this end, we reverse the moves of the segment from (0,0) to $(d + c, d)$, i.e., we take the dual (Feller, 1968, pp.91-92) of this segment, and recombine it with the unchanged portion of P starting at $(d + c, d)$ and ending at (a, b) to form the path P'. It is obvious that P' has $j - 1$ meeting points with the line $y = x$ as the image of the point $(d + c, d)$ of P becomes (0,0) in P' and thus, being the starting point, it will not count as a meeting point. For another path P we get a different P', and each path with $\delta_{a,b}^{(0)} = j - 1$ will be encountered as a P'. Note that this proof can replace the above proof of identity (3). □

We add that there are other simple consequences of the Ballot Theorem. For example, for $m \geq 0$, another application of Theorem 1 yields

$$P(\delta_{n,n+m}^{(-m)} = 1) = \frac{m}{2n + m} = \frac{n + 2m - (n + m)}{n + (n + m)},$$

i.e., the special case of (4) with $a = n, b = a + m, c = m$, and $j = 1$.

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