On the 2-adic order of Stirling numbers of the second kind and their differences

Tamás Lengyel[1]

[1]*Occidental College, 1600 Campus Road, Los Angeles, CA90041, USA*

Let n and k be positive integers, $d(k)$ and $\nu_2(k)$ denote the number of ones in the binary representation of k and the highest power of two dividing k, respectively. De Wannemacker recently proved for the Stirling numbers of the second kind that $\nu_2(S(2^n,k)) = d(k)-1, 1 \leq k \leq 2^n$. Here we prove that $\nu_2(S(c2^n,k)) = d(k)-1, 1 \leq k \leq 2^n$, for any positive integer c. We improve and extend this statement in some special cases. For the difference, we obtain lower bounds on $\nu_2\big(S(c2^{n+1}+u,k)-S(c2^n+u,k)\big)$ for any nonnegative integer u, make a conjecture on the exact order and, for $u=0$, prove part of it when $k \leq 6$, or $k \geq 5$ and $d(k) \leq 2$.

The proofs rely on congruential identities for power series and polynomials related to the Stirling numbers and Bell polynomials, and some divisibility properties.

Keywords: Stirling number of the second kind, congruences for power series and polynomials, divisibility

1 Introduction

The study of p-adic properties of Stirling numbers of the second kind is full with challenging problems. Lengyel (1994) proved that

$$\nu_2(S(2^n,k)) = d(k)-1 \tag{1}$$

for all sufficiently large n, and in fact, $n \geq k-2$ suffices and conjectured that $\nu_2(S(2^n,k)) = d(k)-1$ for all values of $k : 1 \leq k \leq 2^n$. The conjecture was eventually proved by De Wannemacker.

Theorem 1 (De Wannemacker (2005)) *Let $n,k \in \mathbb{N}$ and $1 \leq k \leq 2^n$. Then we have*

$$\nu_2(S(2^n,k)) = d(k)-1. \tag{2}$$

Related results for $k \leq 5$ can be found in Amdeberhan et al. (2008). We generalize De Wannemacker's proof in Section 2. We obtain related results in Section 3. For example, we prove that the 2-adic order of $S(a2^n,b2^n)$ becomes constant as $n \to \infty$ for any positive integers $a \geq b$. As a new direction of investigation, we study the differences of Stirling numbers in Section 4. Lower bounds on $\nu_2\left(S(c2^{n+1}+u,k)-S(c2^n+u,k)\right)$ for any nonnegative integer u and a conjecture on the exact order are presented. For $u=0$, we prove the conjecture provided that $k \leq 6$, or $k \geq 5$ and $d(k) \leq 2$.

The proofs rely on the use of identity (7) by De Wannemacker (2005), the inclusion-exclusion principle based calculation (20) of the Stirling numbers, their generating function (10) and a family of congruential identities for Bell polynomials (23) by Junod (2002). Section 5 utilizes (23) to improve previous results. Section 6 shows that some of the results can be extended to primes other than two.

We note that Flajolet (1982), and Gertsch and Robert (1996) also use formal power series or umbral calculus based techniques to prove divisibility properties.

Exact 2-adic orders are determined in Theorems 2-5, 7, and 12-13. As a summary, we note that the 2-adic order $\nu_2(S(a2^n+u, b2^n+v))$ is discussed with the particular triplet (u, v, b) of parameters. In general, exact values are obtained (except in Remark 2 in which we determine lower bounds on the 2-adic orders). For instance, $(0, 2^m-1, 0)$ (or $(1, 2^m, 0)$), $2 \leq m < \log_2(a2^n+1)$, in Theorem 4 (or in Remark 3); $(0, 2^m, 0), 2 \leq m \leq n$, in Theorem 4; $(u, u, b), 0 \leq u < 2^n$, in Theorem 5; and $(2^m, 0, 1), 0 \leq m \leq n-1$, in Theorem 6; potentially with some other extra assumptions.

In this paper, we include the proofs of the theorems or their sketches if they use generating function or power series based arguments but omit some other proofs.

We note that generating functions (Section 3) and related formal power series (Section 5) based techniques outlined in this paper might lead to improved congruential identities, p-adic results, or their alternative proofs involving other combinatorial quantities, their lacunary series, and their differences, often proved by other methods.

2 A generalization

Theorem 2 *Let $n, k, c \in \mathbb{N}$ and $1 \leq k \leq 2^n$, then*

$$\nu_2(S(c2^n, k)) = d(k) - 1. \tag{3}$$

Remark 1 *In other words, for any fixed $k \geq 1$, we have that $\nu_2(S(c2^n, k)) = d(k) - 1$ if $n \geq \lceil \log_2 k \rceil$. Without loss of generality, we may assume that c is an odd integer (otherwise, we can factor c into a power of two and an odd integer). Note that we obtain*

$$\nu_2(S(4c, 5)) \geq 2 > 1 = d(5) - 1 \tag{4}$$

for $c \geq 1$ odd by (Amdeberhan et al., 2008, formula (3.1))

$$S(n, 5) = \frac{1}{24}(5^{n-1} - 4^n + 2 \cdot 3^n - 2^{n+1} + 1), \quad n \geq 1. \tag{5}$$

For the generalization of (4) see Remark 2. In a similar fashion,

$$S(n, 4) = \frac{1}{6}(4^{n-1} - 3^n + 3 \cdot 2^{n-1} - 1), \quad n \geq 1, \tag{6}$$

proves that $S(c,4)$ is even if c is odd (Amdeberhan et al., 2008, identity (2.14)) while $d(4)-1=0$. Also, $S(c2^n, c2^n-1) = \binom{c2^n}{2}$ and therefore,

$$\nu_2(S(c2^n, c2^n-1)) = n-1 < n \leq d(c2^n-1)-1 = n+d(c)-2,$$

for $n \geq 1, c > 1$ odd. More involved cases of a different type are covered by Theorems 6 and 7. Thus, we cannot expect to extend Theorems 1 and 2 beyond the range $1 \leq k \leq 2^n$, i.e., $\lceil \log_2 k \rceil \leq n$. On the other hand, we mention some extensions in Remark 2.

Proof of Theorem 2: The proof is by induction on $d(c)$. The initial case is with $d(c)=1$, i.e., when $c2^n$ is a power of two, and it is taken care of by Theorem 1.

For $d(c) \geq 2$, we use the identity from (De Wannemacker, 2005, Theorem 2)

$$S(n+m,k) = \sum_{i=0}^{k} \sum_{j=i}^{k} \binom{j}{i} \frac{(k-i)!}{(k-j)!} S(n,k-i) S(m,j) \tag{7}$$

which plays a crucial role in the proof of Theorem 1 in De Wannemacker (2005). Assume that (3) holds for all $c \geq 1$ with $d(c) \leq d-1$ for some $d \geq 2$. We prove that it holds for all c with $d(c)=d$. In fact, let $c'2^n$ be the highest power of two contained in $c2^n$. Then we can write $c2^n$ as the sum $c'2^n + (c-c')2^n$, and by (7) we get that

$$S(c'2^n + (c-c')2^n, k) = \sum_{i=0}^{k} \sum_{j=i}^{k} \binom{j}{i} \frac{(k-i)!}{(k-j)!} S(c'2^n, k-i) S((c-c')2^n, j)$$

since $d(c')=1, d(c-c')=d(c)-1 \leq d-1$, and $k-i, j \leq 2^n$. By the induction hypothesis

$$\nu_2(S(c'2^n, k-i) S((c-c')2^n, j)) = d(k-i) + d(j) - 2,$$

and the proof proceeds exactly the same way as in (De Wannemacker, 2005, Section 3). $\square$

Remark 2 *We can generalize inequality (4) and find that in general, if a is an integer such that $1 \leq a \leq 2^n-2$ then $\nu_2(S(c2^n, 2^n+a)) \geq d(a)+1 > d(a) = d(2^n+a)-1$ for $c \geq 3$ odd. (On the other hand, $\nu_2(S(c2^n, 2^n+a)) = d(a)$ for $a = 2^n-1$, $n \geq 1$ and $c \geq 2$ as we will see in (9) of Theorem 4.) We leave the proof to the reader but note that it is similar to that of Theorems 1 and 2. In fact, after expanding $S(c2^n, 2^n+a) = S((c-1)2^n + 2^n, 2^n+a)$ by identity (7) and focusing on the terms $\binom{j}{i} \frac{(2^n+a-i)!}{(2^n+a-j)!} S((c-1)2^n, 2^n+a-i) S(2^n, j)$, $0 \leq i \leq j \leq 2^n$, the 2-adic order of the terms can now be easily calculated by Theorem 2. It is $\nu_2(\binom{j}{i}) + \nu_2((2^n+a-i)!) - \nu_2((2^n+a-j)!) + \nu_2(S((c-1)2^n, 2^n+a-i)) + \nu_2(S(2^n, j)) \geq 2^n+a-i-d(2^n+a-i)-(2^n+a-j)+d(2^n+a-j)+d(2^n+a-i)-1+d(j)-1 \geq j-i+d(2^n+a)-2 = d(a)-1+j-i$. (Here we used the fact that $d(2^n+a-j)+d(j) \geq d(2^n+a)$.) Now we can combine the terms with 2-adic orders $d(a)-1$ and $d(a)$ to yield the result. By a similar technique, we can also prove that $\nu_2(S(c2^n+b, 2^n+a)) \geq d(a)-2$ for integers $c \geq 3$ odd and $1 \leq b < a < 2^n$. Note that the case with $a=b$ is treated by Theorem 5.*

Note that if c is even then $\nu_2(S(c2^n, 2^n+a)) = d(a)$ for $1 \leq a \leq 2^n - 1$ by Theorem 2. We can further explore the subtle differences between the cases with c odd and even. Numerical experience suggests the following somewhat surprising conjecture.

Conjecture 1 *We have $\nu_2(S((2^r+1)2^n, 2^n+a)) = d(a) + r$ for integers $r \geq 1$, $1 \leq a \leq 2^{n-1}$, and sufficiently large n.*

We also state the following simplified and limited version of the conjecture. It assumes that the 2-adic order $\nu_2(a)$ of a and thus, n are large. We present its proof after that of Theorem 5.

Theorem 3 *We have $\nu_2(S(c2^n, 2^n+a)) = d(a)+\nu_2(c-1)$ for $c \geq 3$ odd, $1 \leq a < 2^n$, if $\nu_2(a)-d(a) > \nu_2(c-1)+1$.*

3 Other properties

Numerical experimentations reveal other interesting properties of the Stirling numbers of the second kind $S(c2^n, k)$. For example, we can slightly improve Theorem 2 for two special values of k.

Theorem 4 *Let $n, c \in \mathbb{N}$ and m be an integer, $2 \leq m \leq n$, then*

$$S(c2^n, 2^m) \equiv 1 \bmod 4 \tag{8}$$

and for $2 \leq m$ with $c2^n > 2^m - 1$,

$$S(c2^n, 2^m - 1) \equiv 3 \cdot 2^{m-1} \bmod 2^{m+1}. \tag{9}$$

Proof of Theorem 4: For $c = 1$ (or any power of two), the proof of (8) is based on that of Theorem 1. For other values of c, the proof is similar to that of Theorem 2.

The proof of congruence (9), however, is rather different. We leave some details to the reader. The cases with $m = 2$ and 3 are easy. For $m \geq 4$, we use the generating function (cf. Comtet (1974))

$$f_k(x) = \sum_{n=0}^{\infty} S(n+k, k)x^n = \frac{1}{(1-x)(1-2x)\cdots(1-kx)} \tag{10}$$

with $k = 2^m - 1$. The proof is based on the formal power series expansion of $f_k(x) \bmod 2^{m+1}$. We note that the coefficient of $x^{c2^n-2^m+1}$ is $S(c2^n, 2^m-1)$. We make two groups of the factors in the denominator. It can be proven that for $m \geq 3$

$$\prod_{i=1}^{2^{m-1}} (1-(2i-1)x) \equiv (1+3x^2)^{2^{m-2}} \bmod 2^{m+1}, \tag{11}$$

and for $m \geq 4$

$$\prod_{i=1}^{2^{m-1}-1} (1-2ix) \equiv 1 + 2^{m-1}x + 2^{m-1}x^2 + 2^m x^4 \bmod 2^{m+1},$$

and thus,

$$\frac{1}{\prod_{i=1}^{2^{m-1}-1}(1-2ix)} \equiv 1 + 3\cdot 2^{m-1}x + 3\cdot 2^{m-1}x^2 + 2^m x^4 \bmod 2^{m+1}. \tag{12}$$

For example, to prove (11), we set $g_{m+1}(x) = \prod_{i=1}^{2^m}(1-(2i-1)x)$. Clearly, $g_3(x) \equiv 1+6x^2+9x^4 \equiv (1+3x^2)^2 \bmod 16$, $g_4(x) \equiv 1+12x^2+22x^4+12x^6+17x^8 \equiv (1+3x^2)^4 \bmod 32$, and note that in general, for $m \geq 2$, $g_{m+1}(x) = \prod_{i=1}^{2^m}(1-(2i-1)x) = g_m(x)\prod_{i=2^{m-1}+1}^{2^m}(1-(2i-1)x) = g_m(x)\prod_{i=1}^{2^{m-1}}(1-(2i-1+2^m)x) \equiv g_m(x)(g_m(x)-h_m(x)) \equiv ((1+3x^2)^{2^{m-2}}+c_1 2^{m+1})((1+3x^2)^{2^{m-2}}+c_1 2^{m+1}-h_m(x)) \equiv (1+3x^2)^{2^{m-1}} \bmod 2^{m+2}$ with some integer c_1 and $h_m(x) = 2^m x g_m(x)\left(\frac{1}{1-x}+\frac{1}{1-3x}+\cdots+\frac{1}{1-(2^m-1)x}\right)$, by induction on m.

Here, we also relied on the fact that, for the power sum $S_j = 1^j+3^j+\cdots+(2^m-1)^j$ we have $\nu_2(S_j) \geq m-1 \geq 2$ for $m \geq 3$, which can be easily proven by induction on m (cf. Lengyel (2007)).

Recall that we need the coefficient of $x^{c2^n-2^m+1}$ in $f_{2^m-1}(x) \bmod 2^{m+1}$. When combined, congruences (11) and (12) give $A \equiv 3\cdot 2^{m-1}(-3)^i\binom{2^{m-2}+i-1}{i} \bmod 2^{m+1}$ with $i = (c2^n-2^m)/2$, making i a multiple of 2^{m-1}. Noting that $(-3)^i \equiv 1 \bmod 2^{m+1}$ and $\binom{2^{m-2}+i-1}{i} \equiv 1 \bmod 4$, this implies that $A \equiv 3\cdot 2^{m-1} \bmod 2^{m+1}$, i.e., the congruence (9). $\square$

Remark 3 *We note that the congruence (9) does not require that the exponent n be at least as large as m but that $c2^n > 2^m-1$, and the proof makes no use of Theorem 2. This congruence allows us to prove that*

$$\nu_2(S(c2^n+1,2^m)) = m-1. \tag{13}$$

In fact, by the usual recurrence $S(c2^n+1,2^m) = 2^m S(c2^n,2^m)+S(c2^n,2^m-1)$ and $\nu_2(S(c2^n,2^m-1)) = m-1$, thus (13) follows.

The above proof of congruence (9) can be modified to yield the following

Theorem 5 *Let a, b, and $n \in \mathbb{N}$, $b \leq a$, and n be sufficiently large (in terms of a and b). Then the 2-adic order of $S(a2^n, b2^n)$ becomes constant as $n \to \infty$. In fact, with $g(a,b) = \nu_2\left(\binom{(2a-b)2^{n-2}-1}{(a-b)2^{n-1}}\right) = d((a-b)2^{n-1})+d(b2^{n-2}-1)-d((2a-b)2^{n-2}-1) = d(a-b)+d(b-1)-d(2a-b-1)$, for any $n > \max\{2, g(a,b)+1\}$ we get that*

$$\nu_2\left(S(a2^n, b2^n)\right) = g(a,b), \tag{14}$$

and in general,

$$\nu_2\left(S(a2^n+u, b2^n+u)\right) = g(a+1,b+1),$$

independently of u, for any integer $u\colon 1 \leq u < 2^n$ as long as $\nu_2(u) > \max\{2, g(a+1,b+1)+1\}$. The periodicity of $g(a,b)$ yields that $\nu_2\left(S((a+2^t)2^n, b2^n)\right) = \nu_2\left(S(a2^n, b2^n)\right)$ if $t \geq \lceil \log_2(2a-b)\rceil$ is a nonnegative integer.

Proof of Theorem 5: We need the coefficient of $x^{(a-b)2^n}$ in $f_{b2^n}(x) \equiv (1+3x^2)^{-b2^{n-2}} \bmod 2^{n-1}$ with $n \geq 3$, since here it is sufficient to combine congruences (11) and (12) $\bmod 2^{n-1}$ rather than $\bmod 2^{n+1}$ for $n \geq 4$. Also note that $\prod_{i=1}^{3}(1-2ix) \equiv 1 \bmod 4$ for $n = 3$. It follows that the 2-adic order of the coefficient is equal to that of $\binom{(2a-b)2^{n-2}-1}{(a-b)2^{n-1}}$, similarly to the proof of (9).

The proof for a general $u > 0$ follows by writing u as $t2^q$ with $q = \nu_2(u) < n$ and some odd $t, 1 \leq t < 2^{n-q}$. Therefore, for example, $a2^n+u = (a2^{n-q}+t)2^q$, and thus, in identity (14), the parameters q,

$a2^{n-q}+t$, and $b2^{n-q}+t$ can play the role of n, a, and b, respectively. In fact, with these values, we get that $g(a2^{n-q}+t, b2^{n-q}+t) = d((a-b)2^{n-q}) + d(b2^{n-q}+t-1) - d((2a-b)2^{n-q}+t-1)$ which simplifies to $d((a-b)2^{n-q})+d(b2^{n-q})-d((2a-b)2^{n-q}) = d(a-b)+d(b)-d(2a-b) = g(a+1, b+1)$. $\square$

Theorem 5 seems to be a powerful tool for tackling the cases with n sufficiently large as is demonstrated in the following proof. Note that the second part of Theorems 6 and 7 can also be handled via this theorem similarly to the

Proof of Theorem 3: We write $a = t2^{n-q}$ with an odd $t : 1 \le t \le 2^{q-1}$ and $1 \le q \le n$. We also write $c = o2^r+1$ with an odd o and $r = \nu_2(c-1) \ge 1$. We set $A = (o2^r+1)2^q$ and $B = 2^q+t$, and apply Theorem 5 by replacing its parameters a, b and n with A, B and $n-q$, respectively. Note that $c2^n = A2^{n-q}$ and $2^n = B2^{n-q}$.

In fact, for a sufficiently large $n-q$ we have $\nu_2\big(S(A2^{n-q}, B2^{n-q})\big) = d(A-B)+d(B-1)-d(2A-B-1) = d(o2^{r+q}-t)+d(2^q+t-1)-d(o2^{r+q+1}+2^{q+1}-2^q-t-1) = (d(o)-1+r+q-d(t)+1)+(1+d(t)-1)-(d(o)+q-d(t)+1-1) = r+d(t) = \nu_2(c-1)+d(a)$. We note that Theorem 5 assumes that $n-q = \nu_2(a) > \max\{2, g(A,B)+1\} = d(a)+\nu_2(c-1)+1$. $\square$

In the next theorem, we obtain a lower bound on $\nu_2(S(c2^n+u, 2^n))$ for any positive integer u. This also extends relation (13) for $m = n$, in some sense. It is worth noting that $\nu_2(S(c2^n, 2^n)) = 0$ has a very different nature.

Theorem 6 *Let $n, u, c \in \mathbb{N}$, then $\nu_2(S(c2^n+u, 2^n)) \ge n-1-\lfloor \log_2 u \rfloor$. If $u = 2^m$ is a power of two, with some integer $m, 0 \le m \le n-1$, then $\nu_2(S(c2^n+2^m, 2^n)) = n-1-m$.*

We note that with the specialization $u = 2^{n-a}, a \ge 1$ integer, we get that $\nu_2(S(c'2^{n-a}, 2^n)) = a-1$ for any integer $c' \ge 2^a$, which includes the fact that $S(c'2^{n-1}, 2^n)$ is odd for $c' \ge 2$.

The previous theorem can be extended to other values to obtain

Theorem 7 *Let $n, k, u, c \in \mathbb{N}, 1 \le k \le 2^n$, and $u \le 2^{\nu_2(k)}$, then $\nu_2(S(c2^n+u, k)) \ge \nu_2(k) - \lfloor \log_2 u \rfloor + d(k) - 2$. Furthermore, if $u = 2^m$ is a power of two, with some integer $m, 0 \le m \le \nu_2(k)-1$, then $\nu_2(S(c2^n+2^m, k)) = \nu_2(k) - m + d(k) - 2$.*

We might as well focus on the tth least significant binary digit of k and obtain the following theorem (which includes the first part of the previous theorem in the special case $t = 1$ which yields that $\nu_2(k) = m_{r-t+1}$).

Theorem 8 *Let $n, k, u, c, t \in \mathbb{N}$, $1 \le k \le 2^n$, $1 \le t \le r = d(k)$, and $u \le 2^{m_{r-t+1}}$ given the binary expansion $k = 2^{m_1} + 2^{m_2} + \cdots + 2^{m_r}$ with $m_1 > m_2 > \cdots > m_r \ge 0$. Then $\nu_2(S(c2^n+u, k)) \ge d(k) - t + m_{r-t+1} - \lfloor \log_2 u \rfloor - 1$.*

Remark 4 *In fact, for a given u, within the scope of this theorem, we can freely pick t as long as $u \le 2^{m_{r-t+1}}$ (thus, it will not apply if $u > k$). Now we find that the largest lower bound on the 2-adic order is achieved at $t = d(k)$, i.e., $\nu_2(S(c2^n+u, k)) \ge m_1 - 1 - \lfloor \log_2 u \rfloor$ for $u \le 2^{m_1}$.*

4 Differences of Stirling numbers

Another interesting property is related to the difference $S(c2^{n+1},k)-S(c2^n,k)$. It appears that its 2-adic order increases by one as n increases by one, provided that n is large enough. As a consequence, this would imply that $\nu_2(S(c2^n,k))$ becomes fix for some large n without explicitly indicating how small this n can be. Of course, Theorem 2 and Remark 1 take care of answering this question. We note that there are some conjectures on the structure of the sets $\{\nu_2(S(c2^n+u,k))\}_{c\geq c_0}$, with c_0 being minimum in order to guarantee $c_0 2^n+u\geq k$, as a function of u for any fixed n and k in Amdeberhan et al. (2008). We state

Conjecture 2 *Let $n,k,a,b\in\mathbb{N}$, $3\leq k\leq 2^n$, and $c\geq 1$ be an odd integer, then*

$$\nu_2(S(c2^{n+1},k)-S(c2^n,k))=n+1-f(k) \tag{15}$$

and

$$\nu_2(S(a2^n,k)-S(b2^n,k))=n+1+\nu_2(a-b)-f(k) \tag{16}$$

for some function $f(k)$ which is independent of n (for any sufficiently large n).

Remark 5 *The cases with $k=1$ and 2 are rather different but trivial. In fact, $S(n_1,1)-S(n_2,1)=0$ for $n_1,n_2\in\mathbb{N}$ and $S(n_1,2)-S(n_2,2)=2^{n_2-1}(2^{n_1-n_2}-1)$ if $n_2<n_1$, thus $\nu_2(S(n_1,2)-S(n_2,2))=n_2-1$. The case with $k=4$ follows by identity (6).*

Remark 6 *To illustrate the above conjecture, we prove a little more for $k=3$. Observe that*

$$S(n,3)=\frac{1}{2}(3^{n-1}-2^n+1),\quad n\geq 1.$$

Let us assume that $a\geq b$. For $n\geq 3$, the Lemma 1 below implies that

$$\nu_2\left(S(a2^n,3)-S(b2^n,3)\right)=-1+\nu_2\left(3^{(a-b)2^n}-1\right)=n+1+\nu_2(a-b),$$

and moreover, for $n\geq 3$ and any nonnegative integer u

$$\nu_2\left(S(a2^n+u,3)-S(b2^n+u,3)\right)=n+1+\nu_2(a-b).$$

It appears that there are only very few exceptions to (15) and (16) requiring the proviso on the large size of n (and perhaps, there is none if we require that $1\leq k\leq 2^{n-1}$). Relations similar to (15) seem to apply to $\nu_2(S(c2^{n+1}+u,k)-S(c2^n+u,k))$ for many nonnegative even integers u (cf. Remark 7 as an illustration to this in a special case).

We are not able to prove Conjecture 2, except for small values of k, e.g., $f(3)=0$ (cf. Remark 6), $f(4)=0, f(5)=2$, and $f(6)=2$ (by evaluating the expressions (20) and (22) using the method in the proofs of Theorems 9 and 10). However, we have the supporting evidence given by Theorem 9 which also suggests that $f(k)\leq\nu_2(k!)-1$ if the conjectured identity (15) holds, and Theorem 11 guarantees the much stronger $f(k)\leq\lceil\log_2 k\rceil-1$. For small values of k, numerical experimentation suggests that

$$f(k)=1+\lceil\log_2 k\rceil-d(k)-\delta(k), \tag{17}$$

with $\delta(4)=2$ and otherwise it is zero except if k is a power of two or one less, in which cases $\delta(k)=1$. This would imply that $f(k)\geq 0$. It appears that $f(2^m)=m-1$ for $m\geq 3$. Note that $\lceil\log_2 k\rceil-d(k)$ is the number of zeros in the binary expansion of k, unless k is a power of two.

Theorem 9 *Let $n, k \in \mathbb{N}, 3 \le k \le 2^n$, u be a nonnegative integer, and $c \ge 1$ be an odd integer, then*

$$\nu_2(S(c2^{n+1}+u,k) - S(c2^n+u,k)) \ge n+2-\nu_2(k!).$$

In the proof we use the following

Lemma 1 *Let $n, m \in \mathbb{N}$, and $c \ge 1$ be an odd integer, then*

$$\nu_2\left((2m+1)^{c2^n}-1\right) = n+2+\nu_2\left(\binom{m+1}{2}\right). \tag{18}$$

Proof of Lemma 1: We factor the expression on the left side of (18):

$$\begin{aligned} (2m+1)^{c2^n}-1 &= \left((2m+1)^{c2^{n-1}}-1\right)\left((2m+1)^{c2^{n-1}}+1\right) \\ &= \left((2m+1)^{2c}-1\right)\prod_{i=1}^{n-1}\left((2m+1)^{c2^i}+1\right). \end{aligned} \tag{19}$$

By the binomial expansion, each factor of the product can be rewritten as

$$(2m+1)^{c2^i}+1 = 1+2m\binom{c2^i}{1}+(2m)^2\binom{c2^i}{2}+\cdots+1 \equiv 2 \bmod 4.$$

This implies that each factor contributes one to the 2-adic order. On the other hand, for the first factor of the last expression in (19), we get that $\nu_2\left((2m+1)^{2c}-1\right) = \nu_2((2m+1)^c-1)+\nu_2\left((2m+1)^c+1\right) = \nu_2(m)+1+\nu_2\left((2m+1)^c+1\right) = \nu_2(m)+1+\nu_2(m+1)+1$ by binomial expansion and $(2m+1)^c+1 = ((2m+1)+1)((2m+1)^{c-1}-(2m+1)^{c-2}+\cdots+1)$. Putting together the factors of (19), the 2-adic order becomes $n+1+\nu_2(m)+\nu_2(m+1)$. The proof is now complete. $\square$

By the well-known identity (cf. Comtet (1974)) for $S(n,k)$

$$k!S(n,k) = \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}i^n$$

it follows that

$$k!\left(S(c2^{n+1},k)-S(c2^n,k)\right) = \sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^{c2^n}\left((k-i)^{c2^n}-1\right). \tag{20}$$

We note that Theorem 9 is the special case of

Theorem 10 *Let $n, k, a, b \in \mathbb{N}$, $3 \le k \le 2^n$, and u be a nonnegative integer, then*

$$\nu_2(S(a2^n+u,k)-S(b2^n+u,k)) \ge n+\nu_2(a-b)+2-\nu_2(k!). \tag{21}$$

Its proof is similar to that of the previous theorem. Assuming that $a \ge b$ we can replace (20) by

$$k!\left(S(a2^n+u,k)-S(b2^n+u,k)\right) = \sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^{b2^n+u}\left((k-i)^{(a-b)2^n}-1\right), \tag{22}$$

and the statement follows by Lemma 1.

5 Towards the proof of the Conjecture 2

We cannot prove Conjecture 2 but we do make some progress in that direction, and at the same time, we improve previously stated results, in general, and for the case when k is a power of two, in particular. We note that for a fixed value of k, the smallest value of n with $1 \le k \le 2^n$ is $\lceil \log_2 k \rceil$, so by Theorem 2, the inequalities $\nu_2(S(c2^{n+1}, k) - S(c2^n, k)) \ge n - \lceil \log_2 k \rceil + d(k)$ and $\nu_2(S(a2^n, k) - S(b2^n, k)) \ge n - \lceil \log_2 k \rceil + d(k)$ hold for this n. Moreover, by Theorem 4 and Remark 6, we have that $\nu_2(S(c2^{n+1}, k) - S(c2^n, k)) \ge n - \lceil \log_2 k \rceil + d(k) + \delta(k) = n + 1 - f(k)$ for this n. This agrees with (17) although in terms of a lower bound rather than the equality in (15).

One possibility for proving Conjecture 2 might be to use differences based on identity (7) or on the congruence by Junod (2002)

$$B_{m+np^\nu}(x) \equiv \sum_{j=0}^{n} \binom{n}{j} (x^p + x^{p^2} + \cdots + x^{p^\nu})^{n-j} B_{m+j}(x) \pmod{\frac{np}{2}\mathbb{Z}_p[x]} \tag{23}$$

with $p = 2$ and proper specializations of the parameters m, n and ν ($m, n \ge 0$ and $\nu \ge 1$ integers), where the Bell polynomials are defined (cf. Junod (2002)) by

$$B_n(x) = \sum_{k=0}^{n} S(n, k) x^k, n \ge 0.$$

We now prove one of our main results, the following weaker version of Conjecture 2, which still improves Theorems 9 and 10 for $k \ge 3$, and it puts us within $d(k) + \delta(k) - 2 < \log_2 k$ of the conjecture (although with some restriction in case of equation (16)).

Note that Theorems 12 and 13 completely prove the conjecture for $k \ge 5$ if $d(k) \le 2$ and $u = 0$. (In this case equation holds in (24).) The cases with $k \le 6$ are taken care of by the comments made on $f(k)$ after Remark 6.

Theorem 11 *Let $n, k \in \mathbb{N}, 3 \le k \le 2^n$, u be a nonnegative integer, and $c \ge 1$ be an odd integer, then*

$$\nu_2(S(c2^{n+1} + u, k) - S(c2^n + u, k)) \ge n - \lceil \log_2 k \rceil + 2. \tag{24}$$

Moreover, let $a, b \in \mathbb{N}$ and $a/2 \le b < a$, then

$$\nu_2(S(a2^n + u, k) - S(b2^n + u, k)) \ge n + \nu_2(a - b) - \lceil \log_2 k \rceil + 2. \tag{25}$$

Proof of Theorem 11: To prove (24), we use (23) with $p = 2, m = u, \nu = 1$, and n replaced by $c2^n$, and consider the coefficients of x^k:

$$\begin{aligned} &S(c2^{n+1} + u, k) \\ &\quad \equiv \textstyle\sum_{j=0}^{c2^n} \binom{c2^n}{j} S(j + u, k - 2(c2^n - j)) \\ &\quad \equiv S(c2^n + u, k) + \textstyle\sum_{j=c2^n - \lceil \frac{k}{2} \rceil + 1}^{c2^n - 1} \binom{c2^n}{j} S(j + u, k - 2(c2^n - j)) \bmod 2^n, \end{aligned} \tag{26}$$

since we observe that $k-2(c2^n-j)>0$ implies that $j>c2^n-\lceil\frac{k}{2}\rceil$. Clearly, in the given range of values $j=c2^n-\lceil\frac{k}{2}\rceil+v, 1\le v<\lceil\frac{k}{2}\rceil\le 2^{n-1}$, we have $\nu_2\big(\binom{c2^n}{j}\big)=\nu_2\big(\binom{c2^n}{\lceil\frac{k}{2}\rceil-v}\big)=n-\nu_2(\lceil\frac{k}{2}\rceil-v)\ge n-(\lceil\log_2 k\rceil-2)$. We note that if $u=0, k\ge 5$, and $d(k)\le 2$ then equality holds in (24) by Theorems 12 and 13.

This proof also applies to (25) with $p=2, m=(2b-a)2^n+u, \nu=1$, and n replaced by $(a-b)2^n$. Again, we consider the coefficients of x^k and get that

$$\begin{aligned}S(a2^n+u,k)&\equiv S(b2^n+u,k)+\textstyle\sum_{j=(a-b)2^n-\lceil\frac{k}{2}\rceil+1}^{(a-b)2^n-1}\binom{(a-b)2^n}{j}\times\\&\quad\times S(j+(2b-a)2^n+u,k-2((a-b)2^n-j)) \bmod 2^{n+\nu_2(a-b)},\end{aligned}$$

and the proof follows as above with $j=(a-b)2^n-\lceil\frac{k}{2}\rceil+v$, $1\le v<\lceil\frac{k}{2}\rceil\le 2^{n-1}$ and $\nu_2\big(\binom{(a-b)2^n}{j}\big)=\nu_2\big(\binom{(a-b)2^n}{\lceil\frac{k}{2}\rceil-v}\big)=n+\nu_2(a-b)-\nu_2(\lceil\frac{k}{2}\rceil-v)\ge n+\nu_2(a-b)-(\lceil\log_2 k\rceil-2)$. Note that $k\le 2^{n+\nu_2(a-b)}$ suffices. $\square$

Now we illustrate a more involved application of (23) to prove equation (15) of Conjecture 2 if $k\ge 8$ is a power of two. (Other powers of two are settled in Remark 5.) We note that this provides a refinement of a direct consequence of equation (8) of Theorem 4.

Theorem 12 *Let $m\ge 3$ be an integer, then*

$$\nu_2(S(2^{m+1},2^m)-S(2^m,2^m))=2, \tag{27}$$

and in general, for an integer $n\ge m\ge 3$ and odd integer $c\ge 1$, we get

$$\nu_2(S(c2^{n+1},2^m)-S(c2^n,2^m))=n-m+2. \tag{28}$$

We mention that Conjecture 2 and equation (17) suggest that $\nu_2(S(c2^{n+1},2^m-1)-S(c2^n,2^m-1))=n+1$ for $n\ge m\ge 2$ and odd $c\ge 1$. Note the striking contrast to (28) in terms of m.

Proof of Theorem 12: To prove identity (27), we use (23) with $p=2, m=0, \nu=1$, and n replaced by 2^m, and consider the coefficients of x^{2^m} in

$$B_{2^{m+1}}(x)\equiv\sum_{j=0}^{2^m}\binom{2^m}{j}x^{2(2^m-j)}B_j(x) \bmod 2^m,$$

i.e., $S(2^{m+1},2^m)\equiv S(2^m,2^m)+\sum_{j=2^{m-1}+1}^{2^m-1}\binom{2^m}{j}S(j,2^m-2(2^m-j)) \bmod 2^m$. The 2-adic order of a general term of the summation with index j, provided that $\nu_2(j)=s<m-1$, is $m-s+\nu_2(S(c'2^s,c'2^{s+1}-2^m))\ge m-s$, with some odd $c'\ge 1$. The smallest such order is $m-(m-2)=2<m$ with the unique $j=3\cdot 2^{m-2}$ (by Theorem 6 with $c=1$, $n=m-1$, and $u=2^{m-2}$). Identity (27) follows.

In general, with $n\ge m$ and $c=1$, we use the above parameters in (23) except that now we replace n by 2^n rather than by 2^m. Similarly to the above proof, it can be shown that $(2^{n-m+2}-1)2^{m-2}=$

$2^n - 2^{m-2}$ is the unique index j that results in a term T with 2-adic valuation as small as $n - m + 2 < n$. In fact, $\nu_2\left(\binom{2^n}{(2^{n-m+2}-1)2^{m-2}}\right) = n - m + 2$, and T is an odd multiple of $2^{n-m+2}S((2^{n-m+2} - 1)2^{m-2}, 2^{m-1})$. This yields (28) by Theorem 6.

The proof with $n \geq m$ and a general odd $c \geq 1$ is similar to the previous case but now n is replaced by $c2^n$. Here $c2^n - 2^{m-2}$ is the unique index j between $c2^n - 2^{m-1} + 1$ and $c2^n - 1$ whose term achieves the smallest valuation $n - m + 2$.

We note that the structure of the 2-adic valuation of the terms shows a remarkably simple pattern. □

Remark 7 *The above proof can be extended to apply to* $\nu_2(S(c2^{n+1}+u, 2^m) - S(c2^n+u, 2^m))$ *if* $u \geq 0$ *is an integer multiple of* 2^{m-2}*, i.e.,*

$$\nu_2(S(c2^{n+1} + d2^{m-2}, 2^m) - S(c2^n + d2^{m-2}, 2^m)) = n - m + 2,$$

for integers $n \geq m \geq 3$*,* $d \geq 0$*, and odd integer* $c \geq 1$*.*

The previous theorem can be modified to yield

Theorem 13 *For integers* $n > m_1 \geq 2$*,* $m_1 > m_2 \geq 0$*, and odd integer* $c \geq 1$*, we get*

$$\nu_2(S(c2^{n+1}, 2^{m_1} + 2^{m_2}) - S(c2^n, 2^{m_1} + 2^{m_2})) = n - m_1 + 1. \tag{29}$$

Proof of Theorem 13: The proof is similar to that of the previous theorem. Here $c2^n - 2^{m_1-1}$ is the unique index j between $c2^n - 2^{m_1-1} - 2^{m_2-1} + 1$ and $c2^n - 1$ whose term achieves the smallest valuation $n - m_1 + 1$. □

6 Other primes

In this paper, we have aimed at divisibility properties by $p = 2$. However, it is worth mentioning that some of the congruences of the previous section can be generalized. For example, for illustrative purposes, we prove the modification of Theorem 11.

Theorem 14 *Let* $p \geq 3$ *be a prime,* $c, n, k \in \mathbb{N}$ *with* $1 \leq k \leq p^n$ *and* $(c, p) = 1$*, and* u *be a nonnegative integer, then*

$$\nu_p(S(cp^{n+1} + u, k) - S(cp^n + u, k)) \geq n - \lceil \log_p k \rceil + 2. \tag{30}$$

Moreover, let $a, b \in \mathbb{N}$ *and* $a/p \leq b < a$*, then*

$$\nu_p(S(ap^n + u, k) - S(bp^n + u, k)) \geq n + \nu_p(a - b) - \lceil \log_p k \rceil + 2. \tag{31}$$

Proof of Theorem 14: We use identity (23) with $m = u, \nu = 1$, the actual prime p, and n replaced by cp^n. We consider the coefficients of x^k:

$$\begin{aligned}
&S(cp^{n+1} + u, k) \\
&\quad \equiv \sum_{j=0}^{cp^n} \binom{cp^n}{j} S\left(j + u, k - p\left(cp^n - j\right)\right) \\
&\quad \equiv S(cp^n + u, k) \\
&\quad + \sum_{j=cp^n - \lceil \frac{k}{p} \rceil + 1}^{cp^n - 1} \binom{cp^n}{j} S\left(j + u, k - p\left(cp^n - j\right)\right) \bmod p^{n+1},
\end{aligned}$$

as we observe that $k - p(cp^n - j) > 0$ implies that $j > cp^n - \lceil \frac{k}{p} \rceil$. Clearly, in the given range of values $j = cp^n - \lceil \frac{k}{p} \rceil + v, 1 \le v < \lceil \frac{k}{p} \rceil \le p^{n-1}$, we have $\nu_p\big(\binom{cp^n}{j}\big) = \nu_p\big(\binom{cp^n}{\lceil \frac{k}{p} \rceil - v}\big) = n - \nu_p(\lceil \frac{k}{p} \rceil - v) \ge n - (\lceil \log_p k \rceil - 2)$.

The proof of inequality (31) is similar to that of (30) and (25). □

We note the relation to some results in Gessel and Lengyel (2001). In fact, Theorem 2 of Gessel and Lengyel (2001) claims that if $u = 0$, c is a multiple of $p - 1$, and k is an odd multiple of p then the lower bound in Theorem 14 can be improved.

Acknowledgements

I would like to thank Victor Moll for sharing their work Amdeberhan et al. (2008) prior to publication and Gregory Tollisen for helpful discussions and comments.

References

Amdeberhan, T., Manna, D., and Moll, V. H. (2008). The 2-adic valuation of Stirling numbers. *Experiment. Math.*, 17(1):69–82.

Comtet, L. (1974). *Advanced combinatorics*. D. Reidel Publishing Co., Dordrecht, enlarged edition. The art of finite and infinite expansions.

De Wannemacker, S. (2005). On 2-adic orders of Stirling numbers of the second kind. *Integers*, 5(1):A21, 7 pp. (electronic).

Flajolet, P. (1982). On congruences and continued fractions for some classical combinatorial quantities. *Discrete Math.*, 41(2):145–153.

Gertsch, A. and Robert, A. M. (1996). Some congruences concerning the Bell numbers. *Bull. Belg. Math. Soc. Simon Stevin*, 3(4):467–475.

Gessel, I. M. and Lengyel, T. (2001). On the order of Stirling numbers and alternating binomial coefficient sums. *Fibonacci Quart.*, 39(5):444–454.

Junod, A. (2002). Congruences pour les polynômes et nombres de Bell. *Bull. Belg. Math. Soc. Simon Stevin*, 9(4):503–509.

Lengyel, T. (1994). On the divisibility by 2 of the Stirling numbers of the second kind. *Fibonacci Quart.*, 32(3):194–201.

Lengyel, T. (2007). On divisibility of some power sums. *Integers*, 7:A41, 6 pp. (electronic).