A COUNTING BASED PROOF OF THE GENERALIZED ZECKENDORF'S THEOREM

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ABSTRACT

We give a counting based short proof of the generalized Zeckendorf's theorem claiming that every positive integer can be uniquely represented as a sum of generalized Fibonacci numbers of order l with no l consecutive indices.

1. INTRODUCTION

Zeckendorf's theorem [4] claims that every positive integer n has a unique representation

$$n = \sum_{i=1}^{m_n} F_{k_i} \text{ such that } 0 \ll k_1 \ll k_2 \ll \cdots \ll k_{m_n}$$

where $a \ll b$ means that $b - a \geq 2$. The usual proof is based on demonstrating by mathematical induction on n, that the greedy algorithm produces the Zeckendorf representation of n, and that it is unique.

The Zeckendorf representation plays an important role in many applications [2]. For example, the losing positions in Wythoff's game are $(a_n, b_n) = (\lfloor n\Phi \rfloor, \lfloor n\Phi^2 \rfloor)$ and $(b_n, a_n), n \geq 1$, and the Zeckendorf representation of the larger coordinate, b_n, can be easily obtained by applying a left shift to that of the smaller one, a_n [1].

A generalized version of Zeckendorf's theorem [2] deals with generalized Fibonacci numbers of order l. We set $G_i = 2^{i-1}, 1 \leq i \leq l$, and define $G_n = \sum_{i=1}^{l} G_{n-i}$ for $n > l$. (The case of $l = 2$ corresponds to the Fibonacci numbers with an index shift.) There is a unique representation for any positive integer n in the form of

$$n = \sum_{i=1}^{m_n} G_{k_i}$$

with no l consecutive indices. Here we present a counting based short proof.

2. PROOF

We call a sum of generalized Fibonacci numbers of order l feasible if it has no l consecutive indices. We accomplish the proof in three steps. First, we prove that there are G_{n+1} feasible sums with terms $G_1, G_2, \ldots, G_n, n \geq l$, then we show that all these feasible sums are bounded from above by $G_{n+1} - 1$, and all of the sums are different. Therefore, every number $i, 1 \leq i \leq G_{n+1} - 1$, is generated exactly once as a feasible sum which concludes the proof.

Let g_n denote the number of feasible sums with terms $G_1, G_2, \ldots, G_n$ (counting possible multiplicities of the sums, though we will see that there are no such occurrences here). We also set $g_0 = 1$. For $n \geq l$, depending on whether the largest non-included term is $G_n, G_{n-1}, \ldots$, or G_{n-l+1}, respectively, we have $g_{n-1} + g_{n-2} + \cdots + g_{n-l}$ possibilities and thus, g_n satisfies the recurrence relation

$$g_n = g_{n-1} + g_{n-2} + \cdots + g_{n-l}.$$

Clearly, $g_i = 2^i = G_{i+1}, 0 \leq i \leq l-1$, which guarantees that $g_n = G_{n+1}, n \geq l$.

Next we show that the sums generated in this way fall between 0 and $G_{n+1} - 1$. This is evident for $n \leq l-1$. For $n \geq l$, the largest sum m includes the terms $G_n, G_{n-1}, \ldots, G_{n-l+2}$ and excludes G_{n-l+1}, then includes $l-1$ consecutive terms if possible, etc. Now simply write G_{n+1} as $G_n + G_{n-1} + \cdots + G_{n-l+1}$ and compare it to m. We can cancel the first $l-1$ terms which leaves us with a comparison of G_{n-l+1} and $G_{n-l} + G_{n-l-1} + \ldots$, and we proceed similarly. Once we reach an index so that $n - j\,l \leq l$, we can see that the difference is at least one in favor of the first quantity.

To prove unicity, suppose that there are two different representations of the same number. We can assume that they have no term in common (otherwise we can remove the terms from both). In this case we can find a term G_i in one representation so that all terms, even the largest G_j in the other representation, are smaller. But by the above argument the second representation results in a sum not exceeding $G_{j+1} - 1$ and thus $G_i - 1$. This contradicts the assumption. $\square$

Note that for the original version of Zeckendorf's theorem $l = 2$ and $G_n = F_{n+1}$. In this case the transfer-matrix method [3] also easily yields g_n but it becomes complicated for larger l. We believe that it will be worth finding a generating function based proof as well.

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