ON CALCULATING THE SPRAGUE-GRUNDY FUNCTION FOR THE GAME EUCLID

Tamás Lengyel*

1. INTRODUCTION

The two-person nim-type game Euclid, E, is played on a board. A position (a, b), or equivalently (b, a), consists of a pair of positive integers. Players alternate moves, a move consisting of decreasing the larger number in the current position by any positive multiple of the smaller number, as long as the result remains positive. The first player unable to make a move loses. Some background information and references on Euclid can be found in [6]. It can be proven that

Theorem A: Player 1 has a winning strategy in E if and only if the ratio of the larger number to the smaller in the starting position is greater than the golden section Φ.

Nivasch [7] gave a polynomial-time algorithm for calculating the Sprague-Grundy function $g(a, b)$ of Euclid for all starting positions (a, b). Fraenkel [3] has recently found an interesting parallel between Euclid and generalized Wythoff games, GW_n. In GW_n, a positive integer n is given, and the two players play in the first quadrant of the integer lattice (with the borders included). The first player starts with two given integers a and b represented by the point (a, b). The moves are of two types: one either subtracts a positive integer from one of the numbers or subtracts $k > 0$ from one and $l > 0$ from the other provided that $|k - l| < n$, while leaving the resulting numbers nonnegative. The player unable to move loses. Note that the case with $n = 1$ is the (standard) Wythoff game. Fraenkel gave polynomial-time algorithms for computing the Sprague-Grundy function. (There is extensive literature on the Wythoff game but we do not rely on or explore this connection any further.)

In the restricted version RE of Euclid, a set Λ of natural numbers is given, and a move decreases the larger number in the current position by some multiple $\lambda \in \Lambda$ of the smaller number, as long as the result remains positive. The game with the special restriction set $\Lambda = \Lambda_k = \{1, 2, \ldots, k\}$, $k \geq 2$, is denoted by RE_k and was analyzed in [6]. Winning strategies and tight bounds on the length of this game assuming optimal play were presented and some extensions were studied. The winner is determined by the parity of the position of the first partial quotient that is different from 1 in a *reduced* form of the continued fraction expansion

*Occidental College, 1600 Campus Road, Los Angeles, CA 90041, USA, e-mail: lengyel@oxy.edu

of b/a (Theorem 5 [6]).

To introduce the reduction, first we take the finite simple continued fraction expansion of $b/a = [a_0, a_1, a_2, \ldots a_n]$. The natural number a_i is called the ith partial quotient (or continued fraction digit) of b/a. If $a_n \geq 2$ then $b/a = [a_0, a_1, a_2, \ldots a_{n-1}, a_n - 1, 1]$ as well, with no change in the digit sum. The former expansion is called the *short* form. Here we always use short forms. In this way, the game E can always be played until some player has a real choice, i.e., $a_i \geq 2$ with some i.

Note that we will apply a slightly modified continued fraction expansion when using the Stern–Brocot tree representation in Section 2.

The reduction of the partial quotients of $[a_0, a_1, \ldots, a_n]$ is motivated by the restriction set Λ_k and results in a reduced sequence of digits after taking the following steps:

- drop any multiple of $k+1$ from the continued fraction expansion and then
- replace every partial quotient $a_i > k$ by $a_i \bmod (k+1)$. (1)

We will see in the new proofs of Theorems A-C why these reductions work. (In [6] we replaced $a_i > 2$ by a 2 if $a_i \equiv 2, 3, \ldots, k \bmod (k+1)$, and for the sake of conformity, we appended a 2 to the end of all reduced sequences not ending in a 2.)

Theorem B (Theorem 5 [6]): Player 1 has a winning strategy for the game RE$_k$, $k \geq 2$, if and only if in the reduced form the first digit that is different from 1 appears at a position with an even index.

For the *misère* version of the games, (i.e., in which the player who makes the final move is the loser) Collins obtained

Theorem C (Theorem 4 [2]): The first player to have a choice can win misère E by adopting the following strategy: when faced with the position $[a_i, a_{i+1}, \ldots, a_n - 1]$, with $a_i \geq 2$, make the same move as in the original version E if at least one of $a_{i+1}, \ldots, a_n - 1 \geq 2$. Otherwise, play so as to leave an odd number of ones (whereas in the original version E one would leave an even number). This strategy works not only for Euclid with no restriction (i.e., E) but restriction sets Λ_k (i.e., RE$_k$), and other equivalent restriction sets.

Equivalent restriction sets will be explained in Section 2. In Section 2, we also introduce and discuss the Stern–Brocot tree representation of E and RE as a general tool to discuss any modifications of Euclid. Section 3 contains the main result: the Sprague-Grundy function for E can be expressed by the beginning segment of the continued fraction expansion of b/a with $a < b$. Combined with the reductions (1), we can calculate the Sprague-Grundy function for RE$_k$ and other equivalent restriction sets.

2. THE STERN-BROCOT TREE AND SUBTRACTION GAMES

The original and restricted versions of Euclid can be analyzed beyond the scope of Theorems A-C too, by using the Sprague-Grundy function. Indeed, we can play the proper sequence of subtraction games on the Stern–Brocot tree ([4] and [1]) and use this fact to determine the Sprague-Grundy function in a *recursive* fashion. We return to other methods of computing the Sprague-Grundy function in Section 3.

The Stern–Brocot tree contains all possible nonnegative fractions (plus the "strange fraction" $\frac{1}{0}$) expressed in lowest terms, with each fraction appearing exactly once. (We can find the Farey sequence F_m for any $m \geq 1$ as a subtree of this tree or view this tree as a structure imposed on the Farey sequence.) The fractions are the nodes of the tree. We put the two "fractions" $\frac{0}{1}$ and $\frac{1}{0}$ at the top level. Then we successively define full levels of the tree by constructing the "mediant" $\frac{a+c}{b+d}$ of two "horizontally" closest entries (from any higher levels) $\frac{a}{b}$ and $\frac{c}{d}$ (on the left and right, respectively) and placing it midway between the two fractions. We connect $\frac{a+c}{b+d}$ to the "fraction" at the immediately preceeding level. (Note that $\frac{1}{1}$ is the only fraction with two predecessors.)

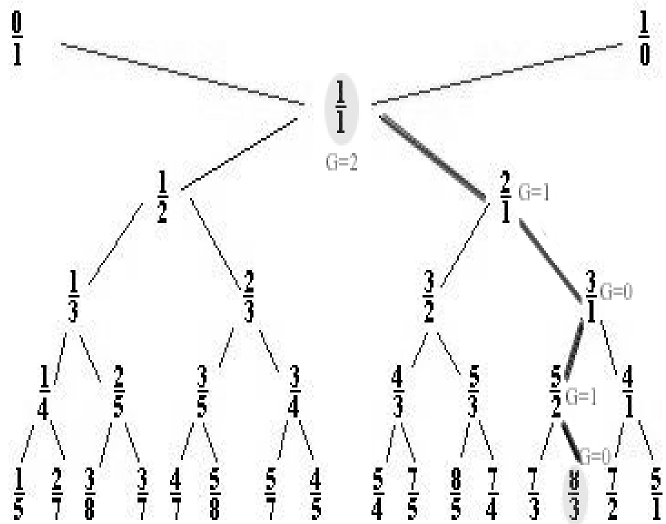

Figure 1: (based on [1]) The top portion of the Stern–Brocot tree

Visually, one can play E and any RE on the Stern–Brocot tree of rationals, or more precisely on the path starting at point $\frac{1}{1}$ and ending at point $\frac{b}{a}$, with $a < b$, corresponding to the Euclid position (a, b). The game is represented by the path formed by intervals of lengths $a_0, a_1, \ldots, a_{n-1}, a_n - 1$, respectively, and corresponding to the partial quotients in the continued fraction expansion of the starting position $b/a = [a_0, a_1, \ldots, a_n], a < b$ (one being removed from the last partial quotient). Two consecutive intervals share one of their endpoints. Starting with the first interval, each move takes the player-to-move to another point of the interval on a lower level.

We also need the notion of subtraction games, and that of Bachet's subtraction game in particular. In the latter case, we start with $n > 0$ chips arranged in a single pile. Players alternate in removing $\lambda \in \Lambda_k$, i.e., $1, 2, \ldots$, or k chips from the pile. The game ends when one of the players takes the last chip and thus wins. In general, in subtraction games the numbers that are subtracted in the consecutive rounds are from some prescribed set, Λ, of positive integers. The Sprague-Grundy function for the game with $\Lambda = \{1, 2, \ldots\}$ is $g(n) = n$ while for Bachet's game is $g(n) \equiv n \bmod (k+1)$, and thus has the period $(0, 1, \ldots, k)$ of length $k+1$. However, we will play subtraction games on a sequence of $n+1$ *connected* intervals as mentioned above, and thus, we will need a slight extension of the Sprague-Grundy functions.

The Sprague-Grundy number $g(a, b)$ can be easily determined. In fact, we assign the Sprague-Grundy value 0 to the terminal node $\frac{b}{a}$, and by moving up on the path, we extend the assignment to all nodes of the segment of length $a_n - 1$ by using the mex operation. We can keep doing this by further extending this process of assignments to the consecutive connected segments of lengths $a_{n-1}, a_{n-2}, \ldots, a_0$. At the points of connections we use the Sprague-Grundy value of the endpoint of the previous segment to start the mex operations. For example, for the Euclid starting position $(3,8)$ we get the path with segments of length $a_0 = 2, a_1 = 1$, and $a_2 - 1 = 1$, in order, as $8/3 = [2,1,2]$. Therefore, the Sprague-Grundy values along the path are $G(\frac{8}{3}) = 0, G(\frac{5}{2}) = 1, G(\frac{3}{1}) = 0, G(\frac{2}{1}) = 1$, and $G(\frac{1}{1}) = 2$. This implies that the Sprague-Grundy value for the starting position $(3,8)$ in E is $g(3,8) = G(\frac{1}{1}) = 2$ which means that Player 1 can win. Indeed, tracing backward, Player 1 *looks ahead* to find a node with a Sprague-Grundy value 0 and moves there: first from $\frac{1}{1}$ to $\frac{3}{1}$, i.e., to $(8-2*3,3) = (2,3)$. This is followed by a forced move to $\frac{5}{2}$, i.e., to $(3-1*2,2) = (1,2)$ by Player 2, and Player 1 wraps up the game by moving to $\frac{8}{3}$, i.e., to $(1, 2-1*1) = (1,1)$. Note that any RE games can be analyzed similarly.

Even in the misère version Player 1 has a winning strategy by moving to $\frac{2}{1}$, i.e., to $(3,5)$.

Note that for other starting points we have to restart the assignments of the values of the G-function. In fact, the value of the Sprague-Grundy function at (a,b), i.e., $g(a,b)$, is tied to the continued fraction expansion of b/a which determines the path from $\frac{1}{1}$ to $\frac{b}{a}$ in the Stern–Brocot tree. However, there is no significance associated with the actual fractions found at the nodes, but only with the respective lengths of the segments of the path.

A recursive algorithm for computing the Sprague-Grundy function for E and RE

Recall that $b/a = [a_0, a_1, a_2, \ldots, a_n]$ but the *path* can be better described by $[a_0, a_1, a_2, \ldots, a_n - 1]$ as we have seen it above. In this section then, we will assume that one has already been removed from the last digit (as in [2]) whenever we consider the Euclid position as $[a_0, a_1, \ldots, a_n]$ to avoid individual discussion involving the last digit. (Therefore, the "*Euclid position representation*" $[a_0, a_1, \ldots, a_n]$ for the game (a,b) differs slightly from the original continued fraction representation of b/a.) To compute the Sprague-Grundy function, we work from right to left: to find the Sprague-Grundy value of $[a_0, a_1, a_2, \ldots, a_n]$, we look first at the Sprague-Grundy value of $[a_n]$, then $[a_{n-1}, a_n]$, and so on. To find $g([a_i, \ldots, a_n])$, we need to know only a_i and the Sprague-Grundy value of the first "connection point" to its right. Therefore, we define the sequence $\{a_i, c_i\}, i = n, n-1, \ldots, 0$, with $c_n = 0, c_{n-1} = g([a_n])$, and in general, $c_i = g([a_{i+1}, \ldots, a_n])$ if $i = n-1, \ldots, 0, -1$. In this way, $c_{i-1} = g([a_i, \ldots, a_n])$ is a function of a_i and c_i, and $g(a,b) = c_{-1}$.

Now we can present fairly simple proofs of Theorems A-C by analyzing the different intervals of the games.

Proof of Theorems A-C. For any game, we observe that if $a_i = 1, i = n, n-1, \ldots, 0$, then

$$c_{i-1} = \begin{cases} 1, & \text{if } c_i = 0, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

In other words, entering a segment of length one means that we are changing winner and loser.

In fact, for E identity (2) can be extended to other values of a_i, and we get that

$$c_{i-1} = \begin{cases} a_i, & \text{if } a_i > c_i, \\ a_i - 1, & \text{if } a_i \le c_i, \end{cases}$$

thus $c_{i-1} > 0$ if $a_i \ge 2$. The proof of Theorem A immediately follows in its equivalent form: the first player facing a digit different from 1 has a winning strategy. Note that $c_{i-1} \ne c_i$.

For RE$_k$, $c_{i-1} = c_i, 0 \le i \le n$, is possible. It happens exactly if $a_i \equiv 0 \bmod (k+1)$, and then we can drop a full segment of length a_i from the path. Also, $g([a_i, a_{i+1}, \ldots, a_n]) = g([1, a_{i+1}, \ldots, a_n])$ if $a_i > 2$ and $a_i \equiv 1 \bmod (k+1)$. In all other cases, i.e., when $a_i \ge 2$ and $a_i \not\equiv 0, 1 \bmod (k+1)$, we have $c_{i-1} \ne 0$, and what matters is which player "faces" this digit a_i during the game. After these reductions we can answer the question whether $c_{-1} = 0$. The answer depends on the parity of the smallest $i \ge 0$ such that the reduced digit is at least two. Now Theorems B and C follow. ∎

We should spend a little more time on some games that are "equivalent" to RE$_k$ for some k, and therefore, can be fully analyzed without posing new difficulties. As a matter of fact, many subtraction-based games with infinite restriction sets display a similar periodicity and are equivalent to Bachet's game for some k [2]. For example, the (single pile) subtraction game with restriction set $\{1, 2, 3, 5, \ldots, p_k, \ldots\}$, where p_k is the kth prime number, has the same Sprague-Grundy function as Bachet's subtraction game with $\Lambda_3 = \{1, 2, 3\}$. More formally, let G be any restriction game and let B$_k$ be Bachet's subtraction game with the set $\Lambda_k = \{1, 2, \ldots, k\}$ of allowed subtractions. We denote the Sprague-Grundy functions for G and B$_k$ by g_{G} and g_{B_k}, respectively, and we write G $\equiv_c$ B$_k$ if games G and B$_k$ have the same Sprague-Grundy function with c ($c \le k$) being the Sprague-Grundy value of the terminal position, i.e., $g_{\mathrm{G}} = g_{\mathrm{B}_k}$ and $g_{\mathrm{G}}(0) = g_{\mathrm{B}_k}(0) = c$.

If G and B$_k$ have the same Sprague-Grundy function for all $c \le k$ then we write G $\equiv$ B$_k$. In the single pile form the value of the terminal position is zero but in Euclid, the position $\{a_i, c_i\}$ is equivalent to the single pile game with a_i counters, the only difference being that the Sprague-Grundy value of the terminal position is c_i rather than zero. The following somewhat surprising theorem shows that G $\equiv_0$ B$_k$ is a necessary and sufficient condition for the equivalence.

Theorem D (Theorem 2 [2]): G $\equiv_0$ B$_k$ if and only if G $\equiv$ B$_k$.

The Stern–Brocot analogy shows that games which are equivalent to one of Bachet's games in the single pile version are equivalent in Euclid as well [2]. Further extensions were given in [2] for move-size restricted versions of Euclid.

3. ON THE SPRAGUE-GRUNDY FUNCTION FOR E AND RE: DIRECT CALCULATIONS

Theorem A tells us that if $a < b$ then $g(a, b) > 0$ for E exactly if $b/a > \Phi$. We can refine this statement and compute $g(a, b)$ for E without the Stern-Brocot correspondence by a *direct* approach with a geometric flavor. In fact, Gabriel Nivasch [7] has recently determined the Sprague-Grundy value $g(a, b)$ of position (a, b) by $g(a, b) = \lfloor b/a - a/b \rfloor$ if $a \le b$, for the unrestricted original version of Euclid, E. Moreover, he illustrates the Sprague-Grundy values in a geometrically appealing form (Figure 2) by observing that the positions $(a, b), a < b$, with Sprague-Grundy value n are those lattice points lying between the rays $b = \Phi_n a$ and $b = \Phi_{n+1} a$, i.e., $\Phi_n < b/a < \Phi_{n+1}$ where $\Phi_n = (n + \sqrt{n^2 + 4})/2, n \ge 1$, and $\Phi_0 = 1$. Note that

$\Phi_1 = \Phi$. Thus, $g(a,b)$ depends on the slope of b/a only. The Sprague-Grundy value of the position $g(a,b)$ with $a > b$ can be determined by symmetry.

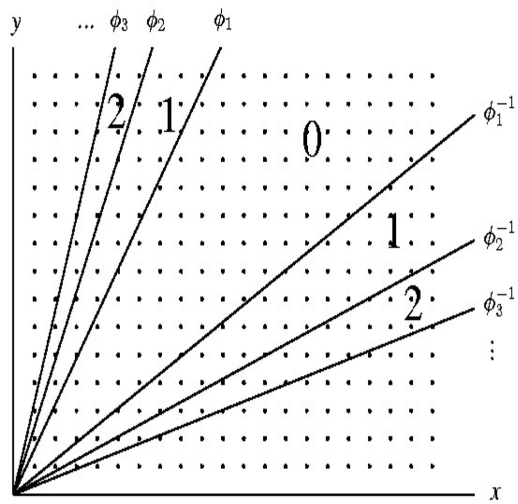

Figure 2: (by Gabriel Nivasch) The Sprague-Grundy function for Euclid, E

Aviezri Fraenkel [3] further developed on Nivasch's observations and computed the Sprague-Grundy function by using a continued fraction expansion based numeration system. From Nivasch's function it also follows that one can express $g(a,b)$ via the continued fraction expansion of b/a in yet another way. We prove the following

Theorem: We take the continued fraction expansion of $b/a = [a_0, \ldots, a_n], a < b$, and append infinity at the end for the classification below to work. The Sprague-Grundy function $g(a,b)$ is equal to k if the continued fraction expansion of b/a starts with

- an even number of identical partial quotients k followed by a larger quotient (case 1) or
- an odd number of identical partial quotients $k+1$ followed by a larger quotient (case 2) or
- an odd number of identical partial quotients k followed by a smaller quotient (case 3) or
- an even number of identical partial quotients $k+1$ followed by a smaller quotient (case 4).

Proof. Fraenkel [3] observed that $\Phi_n = [n, n, \ldots]$. This helps us to prove that for $a < b$

$$\Phi_k < b/a < \Phi_{k+1}, \tag{3}$$

i.e., that $g(a,b) = k$ in all four cases. For example, let us assume that in the continued fraction expansion of b/a we have an odd number of identical partial quotients k followed by a smaller number l (i.e., we are in case 3). It is easy to prove that relation (3) holds true. Indeed,

$[k, k, \ldots, k, l, \ldots] > [k, k, \ldots, k, k, \ldots] = \Phi_k$ is equivalent to $[k, \ldots, k, l, \ldots] < [k, \ldots, k, k, \ldots]$ and hence, to $[\ldots, k, l, \ldots] > [\ldots, k, k, \ldots]$ (with respectively one and two fewer copies of k at the beginning of the expansions). This can be continued to see if $[k, l, \ldots] > [k, k, \ldots]$ holds true, and obviously, it does. Also, $\Phi_{k+1} = [k+1, k+1, \ldots] > [k, k, \ldots, k, l, \ldots]$, so we are done. The other cases are verified similarly. ∎

Note that it does not matter if we use the short form or otherwise. The theorem generalizes the characterization of the winning positions and strategy in E (i.e., Player 1 loses if and only if $k = 0$ and we are in case 2), and makes calculating the Sprague-Grundy function algebraically easy without having to compute the full continued fraction expansion (which is again similar to the above characterization). In this way, we obtain another polynomial-time algorithm to compute the Sprague-Grundy function in terms of the input a and b. Also note that the question of finding a move to a game position with a given Sprague-Grundy value often comes up when playing the sum of games.

For some restricted versions of E, e.g., RE_k, we can use this theorem (with $g(a, b) = k$ if the reduced form is $[\infty]$) on the reduced form (1) of Section 1 to compute the Sprague-Grundy function.

ACKNOWLEDGMENT

I wish to thank Gabriel Nivasch and Aviezri Fraenkel for helpful communication and sharing their results with me. I also thank Greg Tollisen and the referee for careful reading of the manuscript and suggestions.

References

[1] Bogomolny, A. `http://www.cut-the-knot.org/blue/Stern.shtml`, 2004.

[2] Collins, D. "Variations on a theme of Euclid." *Integers, Electronic Journal of Combinatorial Number Theory*, Vol. *5* (2005): #G03, pp. 1-12.

[3] Fraenkel, A. "Euclid and Wythoff Games." Manuscript and personal communication, 2004.

[4] Graham, R. L., Knuth, D. E. and Patashnik, O. Concrete Mathematics. 2nd edition, Reading, MA: Addison-Wesley, 1994.

[5] Khinchin, A. Ya. Continued fractions. New York: Dover, 1977.

[6] Lengyel, T. "A Nim-type Game and Continued Fractions." *The Fibonacci Quarterly*, Vol. *41* (2003): pp. 310–320.

[7] Nivasch, G. "The Sprague-Grundy Function of the Game Euclid." Manuscript and personal communication, 2004.

AMS Classification Numbers: 91A46, 91A20, 11A55, 11A05