ON THE DIVISIBILITY BY 2 OF THE STIRLING NUMBERS OF THE SECOND KIND

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1. Introduction

In this paper we characterize the divisibility by 2 of the Stirling number of the second kind, $S(n,k)$, where n is a sufficiently high power of 2. Let $\nu_2(r)$ denote the highest power of 2 which divides r. We show that there exists a function $L(k)$ such that for all $n \geq L(k)$, $\nu_2\big(k!\,S(2^n,k)\big) = k-1$ hold, independently from n. (Here the independence follows from the periodicity of the Stirling numbers *modulo* any prime power.) For $k \geq 5$, the function $L(k)$ can be chosen so that $L(k) \leq k-2$. We determine $\nu_2\big(k!\,S(2^n+u,k)\big)$ for $k > u \geq 1$, in particular for $u = 1,2,3,$ and 4. We show how to calculate it for negative values, in particular for $u=-1$. The characterization is generalized for $\nu_2\big(k!\,S(c \cdot 2^n + u,k)\big)$ where $c > 0$ denotes an arbitrary odd integer.

2. Preliminaries

The Stirling number of the second kind $S(n,k)$ is the number of partitions of n distinct elements into k non-empty subsets. The classical divisibility properties of the Stirling numbers are usually proved by combinatorial and number theoretical arguments. Here we combine these approaches. Inductive proofs [1] and the generating function method ([11] and [7]) can also be used to prove congruences among combinatorial numbers. We note that Clarke [2] used an application of p-adic integers to obtain results on the divisibility of Stirling numbers.

We define the integer-valued *order* function, $\nu_a(r)$, for all positive integers r and $a > 1$ by $\nu_a(r) = q$, where $a^q | r$, and $a^{q+1} \not| \, r$, i.e., $\nu_a(r)$ denotes the highest power of a which divides r. In this paper we are interested in characterizing $\nu_a(r)$, where $r = k!\,S(n,k)$ and $a = 2$. In [10] we give a lower bound on $\nu_a\big(k!\,S(n,k)\big)$ for $a \geq 3$.

Lundell [11] discussed the divisibility by powers of a prime of the greatest common divisor of the set $\{k!\,S(n,k),\ m \leq k \leq n\}$, for $1 \leq m \leq n$. Other divisibility properties have been found by Nijenhuis and Wilf [12], and recently these results have been improved by Howard [5]. Davis [3] gives a method to determine the highest power of 2 which divides $S(n,5)$, i.e., $\nu_2\big(S(n,5)\big)$. A similar method can be applied for $S(n,6)$ according to Davis.

We will use the well known recurrence relation for $S(n,k)$ which can be proved by the inclusion–exclusion principle

$$k!\,S(n,k) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n. \tag{1}$$

For each prime number p and $1 \leq i \leq p-1$, $i^p \equiv i \pmod p$ by Fermat's theorem, and this implies [1] that, for $2 \leq k \leq p-1$, $S(p,k) \equiv 0 \pmod p$. We note that $S(p,1) = S(p,p) = 1$.

Let $d(k)$ be the sum of the digits in the binary representation of k. Using a lemma by Legendre [9], we get $\nu_2(k!) = k - d(k)$.

Note that, for $1 \leq k \leq 4$, identity (1) implies that $\nu_2\big(S(2^n,k)\big) = d(k) - 1$. By other identities for Stirling numbers (cf. Comtet [1], p. 227), $\nu_2\big(S(2^n,k)\big) = d(k) - 1$ for k, $2^n - 3 \leq k \leq 2^n$.

Classical combinatorial quantities (e.g., factorials, Bell numbers, Fibonacci numbers, etc.) often form sequences that eventually become *periodic* modulo any integer as it was pointed out by I. Gessel. The "vertical" sequence of the Stirling numbers of the second kind, $\{S(n,k) \pmod{p^N}\}_{n \geq 0}$ is periodic, i.e., there exist $n_0 \geq k$ and $\pi \geq 1$ such that $S(n+\pi,k) \equiv S(n,k) \pmod{p^N}$ for $n \geq n_0$.

For $N=1$, the minimum period was given by Nijenhuis and Wilf [12], and this result was extended for $N>1$ by Kwong ([7], Theorems 3.5 and 3.6). From now on $\pi(k;p^N)$ denotes the minimum period of the sequence of Stirling numbers $\{S(n,k)\}_{n\geq k}$ *modulo* p^N, and $n_0(k,p^N)\geq k$ stands for the smallest number of nonrepeating terms. Clearly $n_0(k,p^N)\leq n_0(k,p^{N+1})$. Kwong proved

Theorem A. (Kwong [7]) For $k>\max\{4,p\}$, $\pi(k;p^N)=(p-1)p^{N+b(k)-2}$, where $p^{b(k)-1}<k\leq p^{b(k)}$, i.e., $b(k)=\lceil\log_p k\rceil$.

From now on we assume that $p=2$, $n\geq 1$ and apply Theorem A for this case. Let $g(k)=d(k)+b(k)-2$ and c denote an odd integer. Identity (1) implies $\nu_2\big(S(c\cdot 2^n,k)\big)=d(k)-1$ for $1\leq k\leq\min\{4,c\cdot 2^n\}$. We also set $f(k)=f_c(k)=\max\{g(k),\lceil\log_2\big(n_0(k,2^{d(k)})/c\big)\rceil\}$. Therefore, $c\cdot 2^{f(k)}\geq n_0(k,2^{d(k)})$. We note that $g(k)\leq 2\lceil\log_2 k\rceil-2$. Lemma 3 in [8] yields $f(2^m)=m$ for $m\geq 1$ and $c=1$.

In this paper we prove

Theorem 1. For all positive integers k and n such that $n\geq f(k)$, we have $\nu_2\big(k!\,S(c\cdot 2^n,k)\big)=k-1$ or equivalently, $\nu_2\big(S(c\cdot 2^n,k)\big)=d(k)-1$.

Numerical evidence suggests that the range might be extended for all n provided $2^n\geq k$ and $c=1$. For example, for $k=7$, we get $g(7)=d(7)+b(7)-2=4$ and $n_0(7,2^3)=7$; therefore by Theorem 1, if $n\geq f(7)=4$, then $\nu_2\big(S(2^n,7)\big)=\nu_2\big(S(c\cdot 2^n,7)\big)=2$ for arbitrary positive integer c. Notice, however, that $\nu_2\big(S(8,7)\big)=2$ also. We make the following

Conjecture. For all k and $1\leq k\leq 2^n$, we have $\nu_2\big(S(2^n,k)\big)=d(k)-1$.

By Theorem 1, the Conjecture is true for all $k=2^m$ with $m\leq n$.

In Section 3 we prove Theorem 2, which gives the exact order of $S(n,k)$ in a particular range for k whose size depends on $\nu_2(n)$. Theorem 2 is the key tool in proving Theorem 1. Its proof makes use of the periodicity of the Stirling numbers. It would be interesting to determine the function $L(k)$, which is defined as the smallest integer n' such that $\nu_2\big(S(c\cdot 2^n,k)\big)=d(k)-1$ for all $n\geq n'$. By Theorem 2, we find that $L(k)\leq k-2$ and Theorem 1 improves the upper bound on $L(k)$ if $f(k)<k-2$.

In Section 4 we obtain some consequences of Theorem 2 by extending it for Stirling numbers of the form $S(c\cdot 2^n+u,k)$ where $u=1,2,$ etc. We show how to calculate $\nu_2\big(S(c\cdot 2^n-1,k)\big)$. In neither case does the order of $S(c\cdot 2^n+u,k)$ depend on n (if n is sufficiently large), in agreement with Theorem A.

3. Tools and proofs

We choose an integer l such that $l\leq n$. We shall generalize identity (1) for any modulus of the form 2^l. Observe that, for any i even, $i^n\equiv 0 \pmod{2^l}$, and for all i odd, $(-1)^{k-i}$ will have the same sign as $(-1)^{k-1}$. Therefore, by identity (1)

$$k!\,S(n,k)\equiv(-1)^{k-1}\sum_{\substack{i=1\\ i\text{ odd}}}^{k}\binom{k}{i}i^n \pmod{2^l}. \tag{2}$$

The expression on the right-hand side of congruence (2) is called the *partial Stirling number* [11]. We explore identity (2) with different choices of n in order to find $\nu_2\big(S(n,k)\big)$.

We shall need the following

Theorem 2. Let c be an odd and n be a non-negative integer. If $1\leq k\leq n+2$ then $\nu_2\big(k!\,S(c\cdot 2^n,k)\big)=k-1$, i.e., $\nu_2\big(S(c\cdot 2^n,k)\big)=d(k)-1$.

Roughly speaking, Theorem 2 gives the exact value of $\nu_2\big(k!\,S(m,k)\big)$, for $k \geq 2$, if m is divisible by 2^{k-2}. The higher the power of 2 that divides m, the larger the value of k that can be used. We prove Theorem 1 and then return to the proof of Theorem 2.

Proof of Theorem 1. Without loss of generality, we assume that $k > 4$. Observe that $\nu_2\big(S(c\cdot 2^n,k)\big) = d(k)-1$ is equivalent to

$$S(c\cdot 2^n,k) \equiv 0 \pmod{2^{d(k)-1}} \tag{3}$$

and

$$S(c\cdot 2^n,k) \not\equiv 0 \pmod{2^{d(k)}}. \tag{4}$$

The proof of identities (3) and (4) is by contradiction. To prove the former identity, we set $N = d(k)-1$, hence Theorem A yields

$$\pi(k;2^N) = 2^{d(k)+b(k)-3} \tag{5}$$

where $d(k)+b(k)-3 < g(k) \leq f(k)$.

We assume, to the contrary of the claim, that $S(c\cdot 2^{f(k)},k) \equiv a \not\equiv 0 \pmod{2^N}$. By Theorem A and the period given by (5), we obtain that, for every positive integer $m \geq c$, $S(m\cdot 2^{f(k)},k) \equiv a \not\equiv 0 \pmod{2^N}$. This is a contradiction, for one can select m so that $m\cdot 2^{f(k)}$ becomes $c\cdot 2^n$, with a large exponent n, and by Theorem 2, $S(c\cdot 2^n,k) \equiv 0 \pmod{2^N}$ should be for sufficiently large n. It follows that in fact, $S(c\cdot 2^{f(k)},k) \equiv 0 \pmod{2^N}$, and Theorem A implies $S(c\cdot 2^n,k) \equiv 0 \pmod{2^{d(k)-1}}$ for all $n \geq f(k)$.

To derive identity (4), we set $N = d(k)$. In order to obtain a contradiction, we assume that $S(c\cdot 2^{f(k)},k) \equiv 0 \pmod{2^N}$. Now, by Theorem A, we get $\pi(k;2^N) = 2^{d(k)+b(k)-2}$, where $d(k)+b(k)-2 = g(k) \leq f(k)$. We proceed in a manner similar to that used above by noting that the periodicity now yields $S(m\cdot 2^{f(k)},k) \equiv 0 \pmod{2^N}$ for every positive integer $m \geq c$. It would imply that, for a sufficiently large n, $S(c\cdot 2^n,k) \equiv 0 \pmod{2^{d(k)}}$. However, this congruence contradicts Theorem 2. It follows that $S(c\cdot 2^n,k) \not\equiv 0 \pmod{2^{d(k)}}$ for $n \geq f(k)$, and the proof is now complete. ∎

Proof of Theorem 2. We set $m = c\cdot 2^n$ and select an l such that $1 \leq l \leq n+1$. By Euler's theorem, $\phi(2^l) = 2^{l-1}$; therefore, $i^m \equiv 1 \pmod{2^l}$ if i is odd. By simple summation, identity (2) yields

$$k!\,S(m,k) \equiv (-1)^{k-1} \sum_{\substack{i=1\\ i \text{ odd}}}^{k} \binom{k}{i} = (-2)^{k-1} \pmod{2^l}; \tag{6}$$

therefore, $\nu_2(k!\,S(m,k)) = k-1$, provided $0 \leq k-1 < l$.

We have two cases if $k = n+2$. If m is odd, then $n = 0$ and $k = 2$. The claim is true, since $S(m,2) = 2^{m-1}-1$; therefore, $\nu_2\big(2!\,S(m,2)\big) = 1$. If m is even, then we set $l = n+2 \geq 3$. By induction on $l \geq 3$, we can derive that $i^{2^{l-2}} \equiv 1 \pmod{2^l}$ and identity (6) is verified again. ∎

Remark. By setting $l = n+1$, identity (6) implies the lower bound $\nu_2\big(k!\,S(c\cdot 2^n,k)\big) \geq n+1$, for $k \geq n+2$.

4. Related results

We will use other special cases of identity (2). Similarly to the previous proof, we get that, for all $u \geq 0$, $n \geq l \geq 1$, and $k \leq c \cdot 2^n + u$,

$$(7)\qquad k!\,S(c \cdot 2^n + u, k) \equiv (-1)^{k-1} \sum_{\substack{i=1 \\ i \text{ odd}}}^{k} \binom{k}{i} i^{c \cdot 2^n + u} \equiv (-1)^{k-1} \sum_{\substack{i=1 \\ i \text{ odd}}}^{k} \binom{k}{i} i^u \pmod{2^{l+2}}.$$

We set

$$h(k,u) = (-1)^{k-1} \sum_{\substack{i=1 \\ i \text{ odd}}}^{k} \binom{k}{i} i^u.$$

By identity $x^u = \sum_{j=0}^{u} S(u,j)\binom{x}{j} j!$, we obtain

$$h(k,u) = (-1)^{k-1} \sum_{\substack{i=1 \\ i \text{ odd}}}^{k} \binom{k}{i} \sum_{j=0}^{u} S(u,j)\binom{i}{j} j! = (-1)^{k-1} \sum_{j=0}^{\min\{u,k\}} S(u,j) j! \sum_{\substack{i=1 \\ i \text{ odd}}}^{k} \binom{k}{i}\binom{i}{j}.$$

We focus on the case in which $k > u$ and derive

$$(8)\qquad h(k,u) = (-1)^{k-1} \sum_{j=0}^{u} S(u,j) j! \binom{k}{j} \sum_{\substack{i=j \\ i \text{ odd}}}^{k} \binom{k-j}{i-j} = (-2)^{k-1} \sum_{j=0}^{u} \frac{S(u,j) j!}{2^j} \binom{k}{j}.$$

We introduce the notation $r(k,u) = \nu_2\big(h(k,u)\big)$. Identity (8) implies that $r(k,u) \geq k-u-1$. Observe that $|h(k,0)| = 2^{k-1}$, and for $u \geq 1$,

$$(9)\qquad |h(k,u)|/2^{k-u-1} \leq \sum_{j=1}^{u} j^u 2^{u-j} k^j \leq u(2u)^u (k/2)^u = u(uk)^u.$$

By identity (7), for $u \geq 0$ and any sufficiently large l and $n \geq l$, we have $\nu_2\big(k!\,S(c \cdot 2^n + u, k)\big) = r(k,u)$. In fact, $n \geq l = r(k,u) - 1$ will suffice; for instance, $n \geq k-2$ will be large enough if $u = 0$ (Theorem 2). By identity (9), we derive that $r(k,u) \leq k - u - 1 + u\log_2 k + (u+1)\log_2 u$; therefore, $k - u - 2 + \lceil u \log_2 k + (u+1)\log_2 u \rceil$ can be chosen for n if $u > 0$. We note that, similarly to the proof of Theorem 1, this value might be decreased.

The values of $r(k,u)$ can be calculated by identity (8). For example, if $k > u \geq 0$ then

$$(10)\qquad r(k,u) = \begin{cases} k-1, & \text{if } u = 0 \\ k-2+\nu_2(k), & \text{if } u = 1 \\ k-3+\nu_2(k)+\nu_2(k+1), & \text{if } u = 2 \\ k-4+2\nu_2(k)+\nu_2(k+3), & \text{if } u = 3 \\ k-5+\nu_2(k)+\nu_2(k+1)+\nu_2(k^2+5k-2), & \text{if } u = 4. \end{cases}$$

We state two special cases that can be proved basically differently; although, in the second case, only a partial proof comes out by the applied recurrence relations.

Theorem 3. For $k \geq 2$ and any sufficiently large n, $\nu_2\big(k!\,S(c \cdot 2^n + 1, k)\big) = k - 2 + \nu_2(k)$.

Proof. The proof follows from Theorem 2 and using the recurrence relation $k!\,S(m,k) = k\left\{(k-1)!\,S(m-1,k-1) + k!\,S(m-1,k)\right\}$ with $m = c \cdot 2^n + 1$. Notice, that by Theorem 1, $n \geq \max\{f(k), f(k-1)\}$ will be sufficiently large. ∎

Theorem 4. For $k \geq 3$ and sufficiently large n, $\nu_2\big(k!\,S(c \cdot 2^n + 2, k)\big) = k - 3 + \nu_2(k) + \nu_2(k+1)$.

Proof. By identity (10), we obtain $\nu_2\big(k!\,S(c\cdot 2^n+2,k)\big) = r(k,2) = k-3+\nu_2(k)+\nu_2(k+1)$. Observe that $n \geq \max\{f(k), f(k-1), f(k-2)\}$ suffices. ∎

Notice that we could have used the expansion

$$k!\,S(c\cdot 2^n+2,k) = k\big\{(k-1)!S(c\cdot 2^n+1,k-1)+k!\,S(c\cdot 2^n+1,k)\big\}.$$

By Theorem 3, the first term of the second factor is divisible by a power of 2 with exponent $k-3+\nu_2(k-1)$, while the second term is divisible by 2 at exponent $k-2+\nu_2(k)$. The first factor contributes an additional exponent of $\nu_2(k)$ to the power of 2. We combine the two terms and find that there is always a unique term with the lowest exponent of 2 if $k \not\equiv 3 \pmod 4$. For $k \equiv 3 \pmod 4$, however, this argument falls short and we obtain only the lower bound $k-1$ on $\nu_2\big(k!\,S(c\cdot 2^n+2,k)\big)$.

It turns out that calculating $\nu_2\big(k!\,S(c\cdot 2^n+u,k)\big)$ for negative integers u is more difficult than for positive values. The periodicity guarantees that the order does not depend on n (for sufficiently large n).

We extend the function $h(k,u)$ for negative integers u. We will choose an appropriate value $l \geq 1$ and then set n so that it satisfies the inequality $c\cdot 2^n + u \geq 2^l$. We use the convenient notation $1/i$ for the unique integer solution x of the congruence $i\cdot x \equiv 1 \pmod{2^{l+2}}$ if i is odd. Similarly to identity (7), we obtain

$$k!\,S(c\cdot 2^n+u,k) \equiv (-1)^{k-1}\sum_{\substack{i=1\\ i\text{ odd}}}^{k}\binom{k}{i}\left(\frac{1}{i}\right)^{-u} \pmod{2^{l+2}}. \tag{11}$$

For $u<0$, we set

$$h(k,u) = (-1)^{k-1}\sum_{\substack{i=1\\ i\text{ odd}}}^{k}\binom{k}{i}\left(\frac{1}{i}\right)^{-u}$$

and express $h(k,u)$ as a fraction $\frac{p_k(u)}{q_k(u)}$ in lowest terms. Notice that $\nu_2(p_k(u)) \geq k-d(k)$ holds, since $k!$ divides both sides of (11) for any sufficiently large l. The order of $\nu_2\big(S(c\cdot 2^n+u,k)\big)$ can be determined by choosing $l \geq \nu_2\big(p_k(u)\big)-1$, and the *actual order* is $\nu_2\big(p_k(u)\big)-k+d(k)$. We remark that, for $c=1$, the value of n can be set to $\nu_2(p_k(u))$.

We focus on the case of $u=-1$. Let

$$a_k = \sum_{i=1}^{k}\binom{k}{i}\frac{1}{i}.$$

We get

$$a_s - a_{s-1} - \binom{s}{s}\frac{1}{s} = \sum_{i=1}^{s-1}\frac{1}{i}\left\{\binom{s}{i}-\binom{s-1}{i}\right\} = \sum_{i=1}^{s-1}\frac{1}{s}\binom{s}{i} = \frac{2^s-2}{s} \qquad (s\geq 2).$$

By summation, it follows that $a_k = \sum_{i=1}^{k}\frac{2^i}{i} - \sum_{i=1}^{k}\frac{1}{i}$. Similarly, $b_k = \sum_{i=1}^{k}\binom{k}{i}\frac{(-1)^{i+1}}{i} = \sum_{i=1}^{k}\frac{1}{i}$ (cf. Hietala and Winter [4], or Solution to Problem E3052, in *Amer. Math. Monthly* 94(1987), No. 2, p. 185). Combining these two identities, we obtain

$$h(k,-1) = \sum_{\substack{i=1\\ i\text{ odd}}}^{k}\binom{k}{i}\frac{1}{i} = \frac{1}{2}\sum_{i=1}^{k}\frac{2^i}{i} = \frac{p_k(-1)}{q_k(-1)}. \tag{12}$$

For example, for $k=5$, we get $h(5,-1)=\frac{128}{15}$, $\nu_2\big(p_5(-1)\big)=7$ and $n\geq 7$. E.g., $\nu_2\big(S(127,5)\big)=\nu_2\big(S(255,5)\big)=\cdots=4$. We remark that $\nu_2\big(S(63,5)\big)=4$ holds, too. Notice that the recurrence relation $S(N,K)=K\cdot S(N-$

$1,K)+S(N-1,K-1)$ implies that $\nu_2\big(S(c\cdot 2^n-1,2^m-1)=0$ for every sufficiently large n. By the theory of p-adic numbers [6] and (12), we can derive that, for all sufficiently large n, $\nu_2\big(S(c\cdot 2^n-1,k)\big)=\nu_2\Big(\frac{1}{2}\sum_{i=1}^{k}\frac{2^i}{i}\Big)-k+d(k)=\nu_2\Big(\frac{1}{2}\sum_{i=k+1}^{\infty}\frac{2^i}{i}\Big)-k+d(k\Big)$ where $\nu_2(a/b)$ is defined as $\nu_2(a)-\nu_2(b)$ if a and b are integers. This fact helps us to make observations for some special cases. For instance, if $n>m\geq 3,$ then $\nu_2\big(S(c\cdot 2^n-1,2^m)\big)\geq 2$ holds, and, therefore, $\nu_2\big(S(c\cdot 2^n-1,2^m+1)\big)=1$. Numerical evidence suggests that, for $n>m\geq 4$, $\nu_2\big(S(c\cdot 2^n-1,2^m)\big)=2m-2,$ although we were unable to prove it.

We can determine $\nu_2\big(S(c\cdot 2^n-1,k)\big)$ for most of the odd values of k by systematically evaluating $\nu_2\Big(\sum_{i=1}^{k}\frac{2^i}{i}\Big)$, and obtain

Theorem 5. For all sufficiently large n, $\nu_2\big(S(c\cdot 2^n-1,k)\big)=d(k)-\nu_2(k+1)$, if $k\geq 1$ is odd and $k\not\equiv 5 \pmod 8$ and $k\not\equiv 59 \pmod{64}$ and $k\not\equiv 121 \pmod{128}$.

We leave the details of the proof to the reader.

We note that there is an alternative way of determining $p_k(-1)$. We set

$$I_{k-1}=\frac{k}{2^{k-1}}\frac{1}{2}\sum_{i=1}^{k}\frac{2^i}{i}.$$

One can prove that $I_k=\sum_{j=0}^{k}\frac{1}{\binom{k}{j}}$ and $I_k=\frac{k+1}{2k}I_{k-1}+1$. For other properties of I_k, see Comtet ([1] p. 294, Exercise 15). The latter recurrence relation simplifies the calculation of $\nu_2\big(S(c\cdot 2^n-1,k)\big)$ for large values of k.

We can use identity (7) in a slightly different way and gain information on the structure of the sequence $\{S(c\cdot 2^n+k,k),S(c\cdot 2^n+k+1,k),\cdots,S((c+1)\cdot 2^n+k-1,k) \pmod{2^q}\}$ for every $q,\ 1\leq q\leq d(k)-1$ and sufficiently large n. We observe that the sequence always start with a one and ends with at least $d(k)-q$ zeros. Notice that, for every l and u such that $k>u\geq l>k-d(k)$

$$0=k!\,S(u,k)\equiv(-1)^{k-1}\sum_{\substack{i=1\\ i\text{ odd}}}^{k}\binom{k}{i}i^u \pmod{2^l}.$$

We set $q=l-k+d(k)$. Clearly $1\leq q\leq d(k)-1$. By (7), we get that $k!\,S(c\cdot 2^n+u,k)\equiv 0 \pmod{2^l}$ for all $n\geq l-2\geq 1$. This observation yields that the $d(k)-q$ consecutive terms,

$$S(c\cdot 2^n+u,k) \pmod{2^q},\qquad u=k-d(k)+q,k-d(k)+q+1,\ldots,k-1 \tag{13}$$

are all zeros. Similarly, we can derive that $k!\,S(c\cdot 2^n+k,k)\equiv k!\not\equiv 0 \pmod{2^l}$, i.e., $S(c\cdot 2^n+k,k)\equiv 1 \pmod{2^q}$. Identities (8) and (10) imply that there might be many more zeros in the sequence at and after the term $S(c\cdot 2^n,k) \pmod{2^q}$.

For example, if $k=7$ and $l=5$, then $S(c\cdot 2^n+u,7)\equiv 0 \pmod{2^1}$, for $u=5$ and 6, and all $n\geq 3$. Similarly to the proof of Theorem 1, it follows that identity (13) holds if $n\geq f(k)$. For instance, if $k=23$ and $l=21$, then $S(c\cdot 2^n+u,23)\equiv 0 \pmod{2^2}$ for $u=21$ and 22 provided $n\geq f(23)=7$.

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