DIVISIBILITY PROPERTIES BY MULTISECTION

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1. INTRODUCTION

The p-adic order, $\nu_p(r)$, of r is the exponent of the highest power of a prime p which divides r. We characterize the p-adic order $\nu_p(F_n)$ of the F_n sequence using multisection identities. The method of multisection is a helpful tool in discovering and proving divisibility properties. Here it leads to invariants of the modulo p^2 Fibonacci generating function for $p \neq 5$. The proof relies on some simple results on the periodic structure of the series F_n.

The periodic properties of the Fibonacci and Lucas numbers have been extensively studied (e.g., [13]). (For a general discussion of the modulo m periodicity of integer sequences, see [8].) The smallest positive index n such that $F_n \equiv 0 \pmod p$ is called the rank of apparition (or rank of appearance, or Fibonacci entry-point) of prime p and is denoted by $n(p)$. The notion of rank of apparition $n(m)$ can be extended to arbitrary modulus $m \geq 2$. The order of p in $F_{n(p)}$ will be denoted by $e = e(p) = \nu_p(F_{n(p)}) \geq 1$. Interested readers might consult [6] and [9] for a list of relevant references on the properties of $\nu_p(F_n)$.

The main focus of this paper is the multisection based derivation of some important divisibility properties of F_n (Theorem A) and L_n (Theorem D). A result similar to Theorem A was obtained by Halton [4]. This latter approach expresses the p-adic order of generalized binomial coefficients in terms of the number of "carries." Theorem A can be generalized to include other linear recurrent sequences and a proof without using generating functions was given in Exercise 3.2.2.11 of [6]. The latter approach is implicitly based on multisections.

The generating functions of the Fibonacci and Lucas numbers are

$$f(x) = \sum_{n=0}^{\infty} F_n x^n = \frac{x}{1-x-x^2} \quad \text{and} \quad h(x) = \sum_{n=0}^{\infty} L_n x^n = \frac{2-x}{1-x-x^2},$$

respectively. In this paper the general coefficients of these generating functions will be determined by multisection identities, as we prove

Theorem A [9]: For all $n \geq 0$, we have

$$\nu_2(F_n) = \begin{cases} 0, & \text{if } n \equiv 1, 2 \pmod 3, \\ 1, & \text{if } n \equiv 3 \pmod 6, \\ 3, & \text{if } n \equiv 6 \pmod{12}, \\ \nu_2(n)+2 & \text{if } n \equiv 0 \pmod{12}, \end{cases}$$

$$\nu_5(F_n) = \nu_5(n),$$

$$\nu_p(F_n) = \begin{cases} \nu_p(n) + e(p), & \text{if } n \equiv 0 \pmod{n(p)}, \\ 0, & \text{if } n \not\equiv 0 \pmod{n(p)}, \end{cases} \quad \text{if } p \neq 2 \text{ and } 5.$$

The cases $p = 2$ and $p = 5$ are discussed in Sections 2 and 3, respectively. The general case is completed in Section 4. The case of $p = 2$ has been discussed in [5] using a different approach.

The multisection based technique offers a simplified treatment of this case. We extend the method to the Lucas numbers in Section 5.

By the m-section of a power series $g(x)=\sum_{n=0}^{\infty}a_nx^n$ we mean the extraction of the sum of terms a_lx^l in which l is divisible by m. We use the resulting power series $g_m(x)=\sum_{n=0}^{\infty}a_{mn}x^{mn}$ in its modified form $g_m(x^{1/m})=\sum_{n=0}^{\infty}a_{mn}x^n$ and call it the m-section as well. The corresponding sequence $\{a_{mn}\}_{n=0}^{\infty}$ of coefficients is referred to as the m-section of the sequence $\{a_n\}_{n=0}^{\infty}$. The notion of m-section can be generalized to form a sum of terms with index l ranging over a fixed congruence class of integers modulo m. It will be used in Sections 2 and 5. There are various general multisection identities (cf. [10, p.131] or [1, p. 84]), and they can be helpful in proving divisibility patterns (e.g., [2]). The m-section of the Fibonacci sequence leads to the form

$$\sum_{n=0}^{\infty}F_{mn}x^n=\frac{F_mx}{1-L_mx+(-1)^mx^2}. \tag{1}$$

The denominators are referred to as Lucas factors. For other applications of Lucas factors, see [11].

The present proof of Theorem A is based on a multisection invariant. In fact, we will see in (5), (13), and (14) that $x/(1-x)^2$ or $x/(1+x)^2$ is an invariant of the properly sected Fibonacci generating function taken mod p^2 for every prime $p\neq 5$. The power of p can be improved easily.

We shall need some facts on the location of zeros in the series $\{F_n \bmod m\}_{n\geq 0}$.

Theorem B (Theorem 3 in [13]): The terms for which $F_n\equiv 0 \pmod m$ have subscripts that form a simple arithmetic progression. That is, for some positive integer $d=d(m)$ and for $x=0,1,2,\ldots$, $n=x\cdot d$ gives all n with $F_n\equiv 0 \pmod p$ unless l is a multiple of $n(p)$.

Note that $d(m)$ is exactly $n(m)$, and $d(p^i)=d(p)=n(p)$ for all $1\leq i\leq e(p)$. It also follows that $F_l\not\equiv 0 \pmod p$ unless l is a multiple of $n(p)$.

We denote the *modulo m* period of the Fibonacci series by $\pi(m)$. Gauss proved that the ratio $\frac{\pi(p)}{n(p)}$ is 1, 2, or 4. In fact, we get

Lemma C [9]: The ratio $\frac{\pi(p)}{n(p)}$ can be characterized fully in terms of $x\equiv F_{n(p)-1}\equiv F_{n(p)+1} \pmod p$ by

$$\pi(p)=\begin{cases} n(p), & \text{iff } x\equiv 1 \pmod p,\\ 2n(p), & \text{iff } x\equiv -1 \pmod p,\\ 4n(p), & \text{iff } x^2\equiv -1 \pmod p.\end{cases}$$

In the first case, p must have the form $10l\pm 1$while the third case requires that $p=4l+1$.

We also will repeatedly use two identities (cf. (23) and (24) in [12]) for the Lucas numbers with arbitrary integers $h\geq 0$:

$$L_{2h}=2(-1)^h+5F_h^2, \tag{2}$$

$$L_h^2=4(-1)^h+5F_h^2. \tag{3}$$

It is worth noting that our proofs of Theorems A and D rely on three congruences for the Lucas numbers (cf. Lemmas 1, 2, and 3) which, in turn, can be improved significantly (cf. Lemmas 1′, 2′, and 3′) using the theorems.

2. THE CASE OF $p = 2$

By adding together the six 6-sections $\sum_{n=0}^{\infty} F_{6n+l}x^{6n+l}$, $l = 0, 1, \ldots, 5$, of the generating function $f(x)$, we obtain

$$f(x) = \frac{x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 - 5x^7 + 3x^8 - 2x^9 + x^{10} - x^{11}}{1 - 18x^6 + x^{12}}$$

which is equivalent to the recurrence relation $F_{n+12} = 18F_{n+6} - F_n$, $F_0 = 0$, $F_1 = 1, \ldots, F_{11} = 89$. This immediately implies that

$$\nu_2(F_n) = \begin{cases} 0, & \text{if } n \equiv 1, 2 \pmod 3, \\ 1, & \text{if } n \equiv 3 \pmod 6, \\ 3, & \text{if } n \equiv 6 \pmod{12}. \end{cases}$$

It remains to be proven that

$$\nu_2(F_{12 \cdot n}) = \nu_2(n) + 4. \tag{4}$$

To this end, first we note that

Lemma 1: $L_{12 \cdot 2^k} \equiv 2 \pmod{2^2}$ for all $k \geq 0$.

In fact, the modulo 4 period of F_n is 6, and this implies $L_{6j} \equiv 2F_{6j+1} \equiv 2$ (mod 4) for every integer $j \geq 0$.

By identity (1), we obtain that, for all $k \geq 0$,

$$\sum_{n=0}^{\infty} \frac{F_{12 \cdot 2^k n}}{F_{12 \cdot 2^k}} x^n = \frac{x}{1 - L_{12 \cdot 2^k} x + x^2} \equiv \frac{x}{(1-x)^2} \equiv \sum_{n=1}^{\infty} nx^n \pmod{2^2}. \tag{5}$$

We have $F_{12} = 144 = 2^4 \cdot 9$. By setting $k = 0$ and $n = 2$ in (5) it follows that $F_{12 \cdot 2} / F_{12} \equiv 2$ (mod 2^2), thus $\nu_2(F_{24}) = \nu_2(F_{12}) + 1 = 5$. In general, we use $n = 2$ and observe that

$$\nu_2(F_{12 \cdot 2^{k+1}}) = \nu_2(F_{12 \cdot 2^k}) + 1 = \cdots = \nu_2(F_{12}) + k + 1 = 4 + \nu_2(2^{k+1})$$

follows by a simple inductive argument. We complete the proof of (4) by noting that, for n odd, $\nu_2(F_{12 \cdot 2^k n}) = \nu_2(F_{12 \cdot 2^k})$ holds by (5). □

A sharper version of Lemma 1 can be derived from Theorem A (once it has been proven).

Lemma 1': $L_{12 \cdot 2^k} \equiv 2$ (mod 2^{2k+6}) for all $k \geq 0$.

Proof of Lemma 1': We note that $L_{12 \cdot 2^k} \equiv 2$ (mod 2^{k+3}) can be derived easily from the periodicity of F_n, for $L_{12 \cdot 2^k} \equiv 2F_{12 \cdot 2^k + 1} \equiv 2$ (mod 2^{k+3}) as $\pi(2^l) = 12 \cdot 2^{l-3}$, $l \geq 1$. We notice, however, that the sharper $L_{12} = 322 \equiv 2$ (mod 2^6) also holds. Moreover, identity (2) yields $L_{12 \cdot 2^{k+1}} \equiv 2$ (mod $F_{12 \cdot 2^k}^2$), and we derive that $L_{12 \cdot 2^{k+1}} \equiv 2$ (mod $(2^{4+k})^2$) using Theorem A. Accordingly, we can replace the exponent of p in identity (5). □

3. THE CASE OF $p = 5$

This case is a little more involved. We will find $\nu_5(F_{5^k n})$, $k \geq 1$, in terms of $\nu_5(F_{5^k})$ in three steps. In the first two, we assume that $(n, 5) = 1$, then we deal with the case of $n = 5$.

First, we take the 5-section of $f(x)$ and obtain

$$\sum_{n=0}^{\infty}\frac{F_{5n}}{F_5}x^n=\frac{x}{1-11x-x^2}\equiv\frac{x}{1-x-x^2}\equiv\sum_{n=1}^{\infty}F_nx^n \pmod 5,$$

which guarantees that $\nu_5(F_{5n})=\nu_5(F_5)$ if $(n,5)=1$. In the second step, we try to generalize this relation for indices of the form $5^k n$, $(n,5)=1$, $k\geq 2$. We shall need the following lemma.

Lemma 2: $L_{5^{k+1}}-L_{5^k}\equiv 0 \pmod{25}$ for $k\geq 1$.

Proof of Lemma 2: By identity (3) we have, for $k\geq 1$, that $L^2_{5^{k+1}}-L^2_{5^k}\equiv 0 \pmod{F^2_{5^k}}$. It follows that

$$(L_{5^{k+1}}-L_{5^k})(L_{5^{k+1}}+L_{5^k})\equiv 0 \pmod{25} \tag{6}$$

by Theorem B. Clearly,

$$L_{5^{k+1}}\equiv L_{5^k}\equiv L_5\equiv 1 \pmod 5, \tag{7}$$

thus the factor $L_{5^{k+1}}+L_{5^k}$ cannot be a multiple of 5. Therefore, $L_{5^{k+1}}-L_{5^k}\equiv 0 \pmod{25}$ by identity (6). □

We note that $\nu_5(F_{25})=2$. It is true that, for $k\geq 1$,

$$\sum_{n=0}^{\infty}\left(\frac{F_{5^{k+1}n}}{F_{5^{k+1}}}-\frac{F_{5^k n}}{F_{5^k}}\right)x^n=\frac{x}{1-L_{5^{k+1}}x-x^2}-\frac{x}{1-L_{5^k}x-x^2}$$

$$=(L_{5^{k+1}}-L_{5^k})\frac{x}{1-L_{5^{k+1}}x-x^2}\,\frac{x}{1-L_{5^k}x-x^2}.$$

The first factor is divisible by 25 according to Lemma 2. For $(n,5)=1$, we get

$$\nu_5(F_{5^k n}/F_{5^k})=\nu_5(F_{5^{k-1}n}/F_{5^{k-1}})=\cdots=\nu_5(F_{5n}/F_5)=0, \tag{8}$$

i.e., $\nu_5(F_{5^k n})=\nu_5(F_{5^k})$ by induction on $k\geq 1$.

Now we turn to the case of $n=5$. For $k\geq 1$ and $n=5$, we get that $F_{5^{k+2}}/F_{5^{k+1}}\equiv F_{5^{k+1}}/F_{5^k}$ (mod 25); therefore,

$$\nu_5(F_{5^{k+2}})=\nu_5(F_{5^{k+1}})+1=\cdots=\nu_5(F_5)+k+1.$$

by induction using $\nu_5(F_{25}/F_5)=1$. The proof of the case $p=5$ is now complete. □

Note that, once it is proven, Theorem A guarantees the much stronger lemma.

Lemma 2': $L_{5^{k+1}}\equiv L_{5^k} \pmod{5^{2k}}$ for $k\geq 1$.

We note that an alternative derivation of (8) is possible by (7) but without using Lemma 2:

$$\frac{x}{1-L_{5^{k+1}}x-x^2}\,\frac{x}{1-L_{5^k}x-x^2}\equiv\sum_{n=0}^{\infty}F_n^{(2)}x^n \pmod 5$$

with $F_n^{(2)}$ being the 2-fold convolution of the sequence F_n. The m-fold convolution of the sequence F_n is defined by

$$F_n^{(m)}=\sum_{i_1+i_2+\cdots+i_m=n}F_{i_1}F_{i_2}\ldots F_{i_m},$$

which has the generating function $[f(x)]^m$. Note that, by identity (7.61) on page 354 in [3], $F_n^{(2)} = \frac{1}{5}(2nF_{n+1} - (n+1)F_n) = \frac{n}{5}(2F_{n+1} - F_n) - \frac{1}{5}F_n = \frac{n}{5}L_n - \frac{1}{5}F_n$. We can easily find the period of $F_n^{(m)}$ by the general theory (cf. [8]) or by simple inspection. The latter approach also provides us with the actual elements of the period. It is clear that 100 is the modulo 25 period of $nL_n - F_n$, and $nL_n - F_n$ is divisible by 25 if n is divisible by 5. It follows that $5 | F_n^{(2)}$ if $5 | n$.

4. THE GENERAL CASE

In this section p is a prime different from 2 and 5, and n denotes an integer for which $\nu_p(n)$ is either 0 or 1. We will use either an $n(p)p^k$- or a $2n(p)p^k$-section in obtaining the required divisibility properties. First, we prove

Lemma 3: For any prime $p \equiv 3 \pmod 4$,

$$L_{n(p)p^k} \equiv \begin{cases} 2 \pmod{p^2}, & \text{if } \pi(p)/n(p) = 1, \\ -2 \pmod{p^2}, & \text{if } \pi(p)/n(p) = 2. \end{cases}$$

Proof: Formula (3) yields that, if $h \ge 0$ is even, then $L_{2h}^2 - L_h^2 \equiv 0 \pmod{F_h^2}$. Note that $n(p)$ is even for $p \equiv 3 \pmod 4$ (see [13]). By setting $h = n(p)p^k$ we obtain

$$(L_{2n(p)p^k} - L_{n(p)p^k})(L_{2n(p)p^k} + L_{n(p)p^k}) \equiv 0 \pmod{p^2}. \tag{9}$$

Therefore, either

$$L_{2n(p)p^k} \equiv L_{n(p)p^k} \pmod{p^2} \tag{10}$$

or

$$L_{2n(p)p^k} \equiv -L_{n(p)p^k} \pmod{p^2}, \tag{11}$$

for otherwise both $L_{2n(p)p^k} - L_{n(p)p^k}$ and $L_{2n(p)p^k} + L_{n(p)p^k}$ will be divisible by p. This would lead to $L_{n(p)p^k} \equiv 0 \pmod p$, which is impossible as $L_{n(p)p^k} \equiv 2F_{n(p)p^k+1} \pmod p$. According to identity (2), $L_{2n(p)} = 2 + 5F_{n(p)}^2$, which yields $L_{2n(p)} \equiv 2 \pmod{p^2}$ and also

$$L_{2n(p)p^k} \equiv 2 \pmod{p^2} \tag{12}$$

by Theorem B [13].

If $\pi(p)/n(p) = 1$, then $F_{n(p)+1} \equiv 1 \pmod p$ by Lemma C, and we get $L_{2n(p)} \equiv L_{n(p)} \equiv 2 \pmod p$ and, similarly, $L_{2n(p)p^k} \equiv L_{n(p)p^k} \equiv 2F_{2n(p)p^k+1} \equiv 2 \pmod p$, leading to (10). If $\pi(p)/n(p) = 2$, then $F_{n(p)+1} \equiv -1 \pmod p$ and $L_{2n(p)} \equiv -L_{n(p)} \equiv 2 \pmod p$ and $L_{2n(p)p^k} \equiv -L_{n(p)p^k} \equiv 2 \pmod p$ corresponding to (11). □

We are now able to finish the proof of Theorem A. In the case of $\pi(p)/n(p) = 1$ and 2, we can use

$$\sum_{n=0}^{\infty} \frac{F_{n(p)p^k n}}{F_{n(p)p^k}} x^n = \frac{x}{1 - L_{n(p)p^k}x + x^2} \equiv \frac{x}{(1 \pm x)^2} \equiv \sum_{n=1}^{\infty} (\mp 1)^{n-1} n x^n \pmod{p^2}, \tag{13}$$

which proves $\nu_p(F_{n(p)p^k n}) = \nu_p(F_{n(p)p^k}) + \nu_p(n)$ for $\nu_p(n) \le 1$. In particular, by setting $n = p$, we obtain $\nu_p(F_{n(p)p^{k+1}}) = \nu_p(F_{n(p)p^k}) + 1$, and $\nu_p(F_{n(p)p^{k+1}}) = e(p) + k + 1$ follows by induction on $k \ge 0$. In summary, we derived that $\nu_p(F_{n(p)p^k n}) = e(p) + k + \nu_p(n)$ and the proof is now complete.

On the other hand, if $\pi(p)/n(p)=4$, then we switch from using an $n(p)p^k$-section to a $2n(p)p^k$-section. By the duplication formula (cf. [3] or [12]), we get $F_{2n(p)p^kn}=F_{n(p)p^kn}L_{n(p)p^kn}$ for any integer $n>0$. This yields $\nu_p(F_{2n(p)p^kn})=\nu_p(F_{n(p)p^kn})$. We consider

$$\sum_{n=0}^{\infty}\frac{F_{2n(p)p^kn}}{F_{2n(p)p^k}}x^n=\frac{x}{1-L_{2n(p)p^k}x+x^2}.$$

Identity (12) implies that

$$\sum_{n=0}^{\infty}\frac{F_{2n(p)p^kn}}{F_{2n(p)p^k}}x^n\equiv\frac{x}{(1-x)^2}\equiv\sum_{n=1}^{\infty}nx^n \pmod{p^2}. \tag{14}$$

The proof can be concluded as above for

$$\begin{aligned}\nu_p(F_{n(p)p^kn})&=\nu_p(F_{2n(p)p^kn})=\nu_p(F_{2n(p)})+k+\nu_p(n)\\&=\nu_p(F_{n(p)})+k+\nu_p(n)=e(p)+k+\nu_p(n).\ \square\end{aligned}$$

By means similar to Lemma 1′, we can prove a stronger version of Lemma 3.

Lemma 3′: For any prime $p\equiv 3 \pmod 4$,

$$L_{n(p)p^k}\equiv\begin{cases}2 \pmod{p^{2(k+e(p))}}, & \text{if } \pi(p)/n(p)=1,\\ -2 \pmod{p^{2(k+e(p))}}, & \text{if } \pi(p)/n(p)=2.\end{cases}$$

Proof: We know that $\nu_p(F^2_{n(p)p^k})=2(k+2(p))$ by Theorem A. Thus, we can replace p^2 by $p^{2(k+e(p))}$ in identities (9)-(14). $\square$

We note that, according to Lemmas 1′ and 3′, the denominators of the multisection identities (5), (13), and (14) have either 1 or −1 as a double root modulo some p-power with exponent $2k+6$ or $2(k+2(p))$. This observation, combined with the remarks made in the proofs of the lemmas, helps in obtaining the full description of the structure of the periods of the corresponding multisected sequences [cf. (5), (13), and (14)] with respect to the above-mentioned p-power moduli ($p\neq 5$).

5. LUCAS NUMBERS

By using methods we applied to the Fibonacci sequence, we obtain

$$\sum_{n=0}^{\infty}L_nx^n=\frac{2+x+3x^2+4x^3+7x^4+11x^5-18x^6+11x^7-7x^8+4x^9-3x^{10}+x^{11}}{1-18x^6+x^{12}},$$

which proves that

$$\nu_2(L_n)=\begin{cases}0, & \text{if } n\equiv 1,2 \pmod 3,\\ 2, & \text{if } n\equiv 3 \pmod 6,\\ 1, & \text{if } n\equiv 0 \pmod 6.\end{cases}$$

If $p=5$, then the modulo 5 periodic pattern of L_n is 2, 1, 3, 4, and thus $5\nmid L_n$.

If $p\neq 2$ or 5, then the order $\nu_p(L_n)$ can be derived easily by the duplication formula and Theorem A (see [9]). Here, for the sake of uniformity, we use multisection identities. We need the companion multisection identity to (1) for the Lucas sequence:

$$h_m(x) = \sum_{n=0}^{\infty} L_{mn} x^n = \frac{2 - L_m x}{1 - L_m x + (-1)^m x^2}. \tag{15}$$

As $L_n = F_{2n} / F_n$, we see that L_n is divisible by p only if $2n$ is a multiple of $n(p)$ while n is not; in other words, if n is an odd multiple of $n(p)/2$. This implies that we have to deal only with the case in which $n(p)$ is even. The generalized $\frac{n(p)}{2}$-sected Lucas sequence will suffice to prove

Theorem D: If $p \neq 2$ and $\pi(p)/n(p) \neq 4$, then, for every $k \geq 0$ and $m = (n(p)/2)p^k$,

$$l(x) = \sum_{2 \nmid n} \frac{L_{mn}}{L_m} x^n \equiv \begin{cases} \frac{x(1+x^2)}{(1-x^2)^2} \equiv \sum_{2 \nmid n} n x^n & (\text{mod } p^2), \quad \text{if } \pi(p)/n(p) = 1, \\ \frac{x(1-x^2)}{(1+x^2)^2} \equiv \sum_{2 \nmid n} (-1)^{\frac{n-1}{2}} n x^n & (\text{mod } p^2) \quad \text{if } \pi(p)/n(p) = 2, \end{cases}$$

yielding $\nu_p(L_n) = \nu_p(n) + e(p)$ if $n \equiv n(p)/2 \pmod{n(p)}$.

Proof: Note that the conditions guarantee that $n(p)$ is even. We discuss the case in which $\pi(p)/n(p) = 1$ with $k = 0$ only, while the other cases can be carried out similarly. We note that

$$L_{n(p)/2} l(x) = h_{n(p)/2}(x) - h_{n(p)}(x^2).$$

It is known that $n(p)/2$ is odd if $\pi(p)/n(p) = 1$ (cf. [9]). The common denominator of the above difference can be simplified. In fact, according to identity (15), the denominator of $h_{n(p)}(x^2)$ is

$$1 - L_{n(p)} x^2 + x^4 = 1 - (L_{n(p)/2}^2 + 2)x^2 + x^4$$

by $L_{n(p)} = L_{n(p)/2}^2 - 2(-1)^{n(p)/2}$, which follows from (2) and (3). We get

$$1 - L_{n(p)} x^2 + x^4 = (1 - x^2)^2 - L_{n(p)/2}^2 x^2 \equiv (1 - x^2)^2 \pmod{p^2}.$$

Finally, it is easy to see that $l(x)$ simplifies to

$$\frac{x(1+x^2)}{(1-x^2)^2} \pmod{p^2}. \quad \square$$

The exponent of p can be increased to $2(k + e(p))$ in the above proof and therefore in the theorem also.

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