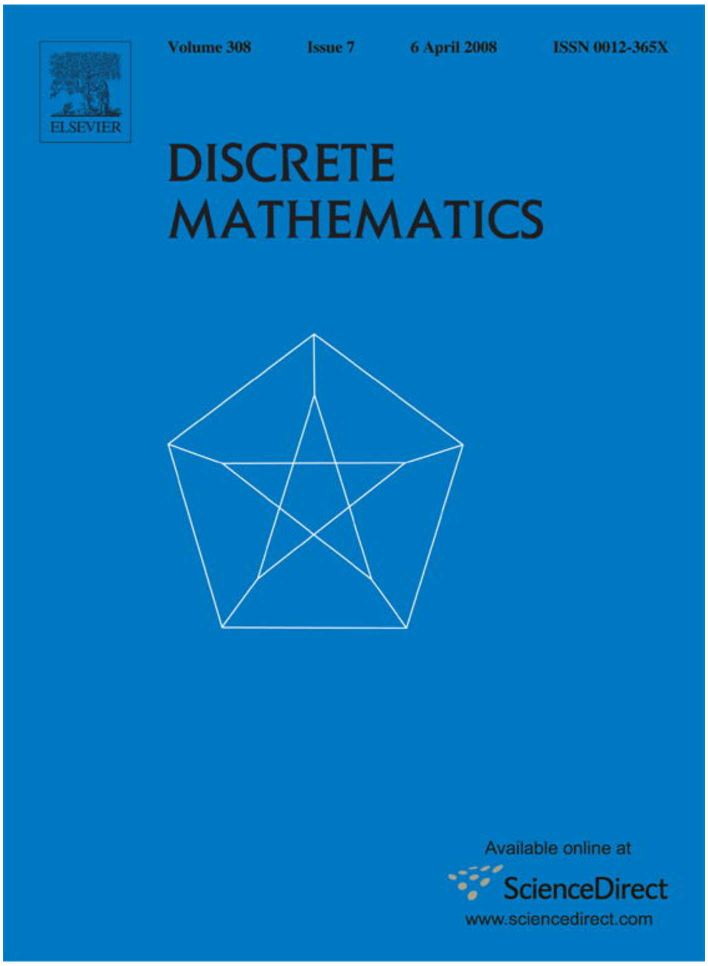
Volume 308
Issue 7
6 April 2008
ISSN 0012-365X
ELSEVIER
DISCRETE MATHEMATICS
Available online at
ScienceDirect
www.sciencedirect.com

Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
Discrete Mathematics 308 (2008) 1130–1136
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

The game of 3-Euclid

David Collins, Tamás Lengyel*

Mathematics Department Occidental College, 1600 Campus Road, Los Angeles, CA 90041, USA

Received 2 February 2006; received in revised form 28 February 2007; accepted 30 March 2007
Available online 6 April 2007

Abstract

In this paper we study 3-Euclid, a modification of the game Euclid to three dimensions. In 3-Euclid, a position is a triplet of positive integers, written as (a, b, c). A legal move is to replace the current position with one in which any integer has been reduced by an integral multiple of some other integer. The only restriction on this subtraction is that the result must stay positive. We solve the game for some special cases and prove two theorems which give some properties of 3-Euclid's Sprague–Grundy function. They provide a structural description of all positions of Sprague–Grundy value g with two numbers fixed. We state a theorem which establishes a periodicity in the P positions (i.e., those of Sprague–Grundy value $g = 0$), and extend some results to the misère version.

Keywords: Impartial games; Game Euclid; Sprague–Grundy function; Misère version

1. Introduction

The game of 3-Euclid is a modification of the game Euclid. Apparently, Euclid was introduced by Cole and Davie [1]. The interested reader can find more details and developments in Spitznagel [10], Grossman [5], and Straffin [11]. Some aspects and generalizations of Euclid were discussed in Lengyel [7], Collins [3], Fraenkel [4], and Lengyel [8]. The Sprague–Grundy function of Euclid and misère Euclid was found by Nivasch [9] and Gurvich [6], respectively.

In 3-Euclid, a position is a triplet of positive integers, written as (a, b, c). Since the order of the integers is irrelevant, we shall assume that $a \leqslant b \leqslant c$, with the exception that sometimes c is selected later without knowing its magnitude. In this case, the (a, b, c) notation means only that $a \leqslant b$.

A legal move is to replace the current position with one in which any integer has been reduced by an integral multiple of some other integer. The only restriction on this subtraction is that the result must stay positive. The game ends when there are no more legal moves; this occurs exactly when the three numbers are equal. In the normal game, the last player to move wins.

We shall refer to positions in which the previous player to move will win with optimal play as P positions, and write $(a, b, c) \in \mathrm{P}$. (The corresponding move to such a position is often referred to as a winning move.) Positions from which the next player to move can win are N positions, written as $(a, b, c) \in \mathrm{N}$. There are three distinct types of moves in 3-Euclid. Moves which are a subtraction of a multiple of the smallest integer from the largest (i.e., of a multiple of a from c), we refer to as 1–3 moves. Analogously, subtractions of a multiple of b from c are 2–3 moves and subtractions of a multiple of a from b are 1–2 moves.

* Corresponding author.
E-mail address: lengyel@oxy.edu (T. Lengyel).

0012-365X/$ - see front matter © 2007 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2007.03.064

In Section 2 we discuss some special cases, while Section 3 contains one of our main results, Theorem 2, on the number of x values for which $(a, b, x) \in \mathrm{P}$. We give an upper bound on x in Section 4. Theorem 2 is extended in Section 5 to Sprague–Grundy values different from zero (Theorem 4). The last section is devoted to periodicity issues and the misère version of 3-Euclid.

2. Special cases

For positions (a, b, c) with small a-value, we can determine the winner quite easily. Some other special cases are also of interest. Although some techniques of these special cases do not generalize, the results are used in later theorems.

Facts. (1) *For $a = 1$, $(1, b, c) \in \mathrm{P}$ if and only if $b = c$. This is easily seen. If $b \neq c$, the player to move can subtract $c - b$ from c to move to $(1, b, b)$. On the other hand, in a position of form $(1, b, b)$, any move will leave a position $(1, b - k, b)$ with $b - k \neq b$. The 3-Euclid position $(1, b, c)$ is equivalent to the game of Nim with two piles of sizes $b - 1$ and $c - 1$.*

(2) *For $a = 2$, $(2, b, c) \in \mathrm{P}$ if and only if the position is of the one of the forms $(2, 2k - 1, 2k)$ with $k \geqslant 2$ or $(2, b, b)$ with $b \geqslant 2$. Consider first a position of the form $(2, 2k - 1, 2k)$. Then, the 2–3 move of reducing $2k$ by $2k - 1$ results in an N position by the first fact above. On the other hand, a 1–2 or 1–3 move of removing $2m$, $m \geqslant 1$, from either $2k - 1$ or $2k$ allows the second player to return to another P position (either $(1, 2, 2)$ or $(2, 2k' - 1, 2k')$) by removing $2m$ from whichever of $2k - 1$ or $2k$ was not reduced by the first player.*

In a position of the form $(2, b, b)$, the first player must move to a position of the form $(2, b - 2m, b)$ after removing a multiple of 2 from b. If $b - 2m \geqslant 2$, the second player can mirror the first player's move and play to $(2, b - 2m, b - 2m)$, winning. If $b - 2m = 1$, the second player can use a 1–3 move to reach the P position $(1, 2, 2)$. It is easily seen that a position of form $(2, 2k - 1, 2k)$ or $(2, b, b)$ can be reached from any position $(2, b, c)$ not already of one of those forms.

(3) *The position $(a, a + 1, a + 2) \in \mathrm{N}$ if and only if a is odd. The 1–2 and 2–3 moves leave an N position by (1) above. The 1–3 move to $(2, a, a + 1)$ wins if and only if a is odd by fact (2).*

(4) *If $d = \gcd(a, b, c) > 1$ then the position (a, b, c) is equivalent to $(a/d, b/d, c/d)$. For example, if a divides b then $(a, b, b) \in \mathrm{P}$ immediately follows as $(1, b/a, b/a) \in \mathrm{P}$ by fact (1).*

The special case in which two integers are equal is also interesting. The following innocent-looking theorem's proof is far more involved than one might expect.

Theorem 1. *The 3-Euclid position $(a, b, b) \in \mathrm{P}$, for all $a \leqslant b$.*

Proof. According to fact (4), we can assume that $\gcd(a, b) = 1$. The proof is obvious for $a = 1$ by fact (1). Thus, we can assume that $1 < a < b$. □

We give an existence proof only, showing that a winning strategy exists for the second player, but not exhibiting that strategy. In the proof, we employ a strategy-stealing technique suitable to a nonconstructive proof, known as the *principle of dorminess* (the name is suggested by a vague resemblance between our technique and one used by Spades players to force their opponents into helping them make a bid).

Lemma 1 (*Principle of dorminess*). *If from position (a, b, c) every 1–2 or 2–3 move loses (i.e., reaches an N position), then $(a, b, c + sa) \in N$ for all $s \geqslant 1$.*

To see why, consider what happens if the first player moves from $(a, b, c + sa)$ to (a, b, c). We have assumed that 1–2 or 2–3 replies lose for the second player. If on the other hand the second player makes another 1–3 move to $(a, b, c - ta) \in \mathrm{P}$ for some $t \geqslant 1$ (and this move is unique if it exists at all), then the resulting position could have been reached by the first player from the original position $(a, b, c + sa)$ directly had she removed a greater multiple of a from c. In short, $(a, b, c - ta) \in \mathrm{P}$ for some $t \geqslant 0$, and the first player either (1) did move there (by playing to (a, b, c)), or (2) could have (by subtracting a larger multiple). In either case, $(a, b, c + sa) \in \mathrm{N}$, as desired. The move from $(a, b, c + sa)$ to (a, b, c) is referred to as a *dormy* move. Provided we are interested in showing only that $(a, b, c + sa) \in \mathrm{N}$, and not in exhibiting the winning strategy, it is sufficient to exhibit a dormy move. We shall employ this technique at several points in the proof.

The theorem follows from Lemmas 2 and 3.

Lemma 2. *From a 3-Euclid position (a, b, b), $a < b$, facing the first player, the second player has a strategy which leads to a position (a', b', c') with $a' \equiv c' \bmod b'$ and the second player to play.*

Proof of Lemma 2. From (a, b, b), the first player must move to a position of form $(b - ka, a, b)$ or $(a, b - ka, b)$. If $b - ka < a$, we make the substitutions $a' = b - ka$, $b' = a$ and write the position as $(a', b', a' + kb')$, and we have $a' \equiv a' + kb' \bmod b'$, and it is the second player's turn.

If, on the other hand, we have $b - ka \geqslant a$, the second player can move to $(a, b - ka, b - ka)$, and by continuing to mirror the first player's moves in this way, the second player guarantees that her opponent will eventually be forced to move below a. □

Lemma 3. *For all $n \geqslant 1$, we have that the position $(a, b, nb + a) \in$ N, and it has a winning 1–3 move.*

Proof of Lemma 3. Let $n \geqslant 1$ and assume by induction that the statement is true for all positions reachable from $(a, b, nb + a)$. We will prove now that the statement is also true for $(a, b, nb + a)$.

First, if $n = 1$, the first player can make the 1–3 move from $(a, b, b + a)$ to (a, b, b), and by Lemma 2 is guaranteed to be on move in a smaller position $(a', b', n'b' + a')$. By our assumption $(a', b', n'b' + a') \in$ N; thus, $(a, b, b + a) \in$ N, and the first player can make a winning 1–3 move in $(a, b, b + a)$.

Now suppose that $n \geqslant 2$. Consider the $n - 1$ 1–3 moves to $(a, b, nb - (n - 1)a)$, $(a, b, nb - (n - 2)a), \ldots,$ $(a, b, nb - ka), \ldots, (a, b, nb - a)$ by the first player. Let M denote the set of these positions, i.e.,

$$M = \{(a, b, nb - ka), 1 \leqslant k \leqslant n - 1\}.$$

Note that we have $nb - ka \geqslant nb - (n - 1)a > b$.

We shall show below that in (at least) one position of M, there is no winning 2–3 or 1–2 move for the second player. Hence, one of the moves in M is dormy, and it follows immediately that $(a, b, a + nb) \in$ N.

2–3 *moves*: Each of the $n - 2$ replies of removing $b, 2b, \ldots, (n - 2)b$ from $nb - ka$ by the second player can win in the position $(a, b, nb - ka)$ for only one value of k by the usual argument. (For, else there would exist positions $(a, b, (n - c)b - ka) \in$ P and $(a, b, (n - c)b - k'a) \in$ P. But we could move from one of these positions to the other by subtracting a multiple of a, a contradiction.) Clearly, subtracting nb from $nb - ka$ yields a negative result for all $1 \leqslant k \leqslant n - 1$. We now show that removing $(n - 1)b$ cannot win for any k value either.

The result of such a subtraction is either a position of the form $(b - ka, a, b)$ or $(a, b - ka, b)$. In the first case $b - ka \equiv b \bmod a$, so the position is in N by our assumption. In the second case, the first player can now move to $(a, b - ka, b - ka)$, and is guaranteed to reach a winning position (as at one point the first player can make sure that she is to move from a position $(a', b', c') \in$ N with $a' \equiv c' \bmod b'$ by Lemma 2 and our assumption).

We conclude that for the second player there are not enough 2–3 moves to win in all of our $n - 1$ positions of M.

1–2 *moves*: To complete the proof, we need only to show that a 1–2 move by the second player cannot win in any position of M. From $(a, b, nb - ka)$, a 1–2 move leaves a position of form $(a, b - ca, nb - ka)$ or $(b - ca, a, nb - ka)$, $1 \leqslant c < b/a$, for the first player. Note that the first and second numbers cannot be equal for $\gcd(a, b) = 1$.

Consider the former case. From $(a, b - ca, nb - ka)$, the first player can move to $(a, b - ca, nb - (nc - 1)a)$ (case (i)), unless the position is already of this form (case(ii)), since $nc - 1 \geqslant n - 1 \geqslant k$. Furthermore, we have $nb - (nc - 1)a > b - (c - 1)a > b - ca$. In both cases, since $a \equiv nb - (nc - 1)a (\bmod\ b - ca)$, our assumption guarantees that there exists a winning 1–3 move from $(a, b - ca, nb - (nc - 1)a)$. In case (ii), it is for the first player, and the proof is done. In case (i), it is for the second player. But the first player can make any such move directly from $(a, b - ca, nb - ka)$ by removing a suitably large multiple of a from $nb - ka$; so $(a, b - ca, nb - ka) \in$ N.

In the position $(b - ca, a, nb - ka)$, the first player can make a 1–3 move—by taking $n - 1$ times the first number from the third number—to $(b - ca, a, b - (k - (n - 1)c)a) = (b - ca, a, b - (k + c - nc)a)$, and we have $b - (k + c - nc)a \geqslant b - ((n - 1) + 1 - n)a = b > a$. Since $b - ca \equiv b - (k + c - nc)a \bmod a$, our assumption guarantees that a winning 1–3 move from $(b - ca, a, b - (k + c - nc)a)$ exists; and such a move is one which the first player could have made directly from $(b - ca, a, nb - ka)$. Thus, $(b - ca, a, nb - ka) \in$ N.

This shows that a 1–2 move can never win from any of the $n-1$ positions of M, completing our proof of Lemma 3. $\square$

The proof of Theorem 1 is now complete. $\square$

Corollary. *The 3-Euclid position with two equal numbers is a P position if and only if the two larger numbers are equal.*

3. A 3-Euclid theorem

We now establish a result which generalizes a pattern observed in the special cases $a = 1$ and $a = 2$, and extends the facts (1)–(2) of Section 2.

Theorem 2. *In the game of 3-Euclid, the position $(a, b, x) \in \mathrm{P}$ for exactly a values of x, where a and b are constants with $a \leqslant b$ and x is a variable with $x \geqslant 1$.*

Proof. We show that there is exactly one x-value for which $(a, b, x) \in \mathrm{P}$ in each congruence class mod a. It follows, of course, that there are exactly a x-values total. We note first that it is easily seen that if $(a, b, x) \in \mathrm{P}$ and $(a, b, x') \in \mathrm{P}$, then $x \not\equiv x' \bmod a$. For if they were, we could move from one P position to another by removing a multiple of a from the larger of x and x', a contradiction. Hence, it is sufficient to show that there is at least one x-value congruent to $m \bmod a$, for all $0 \leqslant m < a$.

Suppose to the contrary that $(a, b, x) \in \mathrm{N}$, for all $x \equiv m \bmod a$, for an arbitrary m with $0 \leqslant m < a$; thus, there must be a move to a P position from any such (a, b, x).

The proof has two cases, $a = b$ and $b > a$. Assume first that $b > a$. Define p as follows: $p = \lceil b/a \rceil - 1$ (p denotes the number of 1–2 legal moves from the position (a, b, x)). Choose k such that $k \equiv m \bmod a$ and $b \leqslant k < b + a$. Let t be the least integer such that $k + (p + t - 1)a - tb \leqslant 0$. Such an integer is guaranteed to exist since $b > a$, and in fact

$$t = \left\lceil \frac{k + (p-1)a}{b - a} \right\rceil. \tag{1}$$

Consider now the sequence S of $p + t$ positions $(a, b, k), (a, b, k + a), (a, b, k + 2a), \ldots, (a, b, k + (p + t - 1)a)$. Note that there is a move between any pair in S. In each of these $p+t$ positions, there are at most $p+t-1$ moves which are not 1–3 moves, namely, the p possible 1–2 moves of removing $a, 2a, \ldots, pa$ from b and the $t - 1$ possible 2–3 moves of removing $b, 2b, \ldots, (t - 1)b$ from (the actual) x. By definition, it is impossible to remove a larger multiple of a from b or b from (the actual) x, and it is irrelevant that some of the 2–3 moves are impossible in some positions in the sequence S. Each of these $p + t - 1$ moves can be a winning move (i.e., a move to a P position) in at most one of the $p + t$ positions in the sequence, as noted in the first paragraph of the proof. So, by the pigeonhole principle, there is a position in the sequence for which the winning move is not a 1–2 or 2–3 move. So it is a 1–3 move, i.e., a removal of a multiple of a from x. But this is a contradiction, since then the result is a position $(a, b, x') \in \mathrm{P}$, for which $x' \equiv x \equiv m \bmod a$, contrary to our assumption that no such positions exist.

For the case $a = b$, let m be an integer with $0 \leqslant m < a$. By Theorem 1 we immediately have $(m, a, a) \in \mathrm{P}$, and our proof is complete. We note that this case can be proven without Theorem 1. For, consider the position $(a, a, a + m)$. If it is not a P position, then $(m, a, a) \in \mathrm{P}$, for that is the only legal move from $(a, a, a + m)$. Hence, $(a, a, x) \in \mathrm{P}$ for either $x = m$ or $x = a + m$; thus, for some $x \equiv m \bmod a$, $(a, b, x) \in \mathrm{P}$. $\square$

Note that trivially, there are at most three winning moves from any position by the reasoning of the proof. The position (a, b, c) with three winning moves and $a + b + c$ minimized is (6, 13, 23).

4. An upper bound

If $a = b$ then we have $(x, b, b) \in \mathrm{P}$ for all x, $1 \leqslant x \leqslant b - 1$, by Theorem 2. For $a < b$, we can use the proof of Theorem 2 to find an upper bound, albeit not a very useful one, on the x-value for which (a, b, x) is a P position. All we need to

a=3	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1			P															
2			P															
3	P	P	P															
4		P	P						P									
5					P	P	P											
6					P	P	P											
7					P	P	P											
8								P	P	P								
9				P				P	P									
10								P		P		P						
11											P	P	P					
12										P	P	P						
13											P		P		P			
14														P	P	P		
15													P	P	P			
16														P		P		P
17																	P	P
18																P	P	P

Fig. 1. The P positions of 3-Euclid for $a = 3$, $1 \leqslant b, c \leqslant 18$.

do is to find the largest possible value of $k + (p + t - 1)a$. Maximizing k increases both k and t without affecting p, so we assign to k the value $b + a - 1$, the largest it can attain. By (1) we have

$$t = \left\lceil \frac{b - 1 + (\lceil b/a \rceil - 1)a}{b - a} \right\rceil.$$

Thus, we get

Theorem 3. *If $a < b$ then the following upper bound holds for the x-value in $(a, b, x) \in$ P:*

$$x \leqslant \left(\left\lceil \frac{b - 1 + (\lceil b/a \rceil - 1)a}{b - a} \right\rceil + \lceil b/a \rceil - 1 \right) a + b - 1.$$

Some simplifications lead to the simple but weaker upper bound $x < 3a + 2b$ for $b > 3a$. One of the referees pointed out that Theorem 5 implies the stronger bound $x < b + k_a$ for some constant k_a that depends on a (cf. Fig. 1 with all P positions for $a = 3$, $1 \leqslant b, c \leqslant 18$).

Example. Consider the position (5, 12, 51). Plugging $a = 5$, $b = 12$ into our formula, we get 36 as an upper bound on x. Since $51 > 36$, we conclude that $(5, 12, 51) \in$ N. In fact, for the Sprague–Grundy values we get that $g(5, 12, 13) = 0$ and $g(5, 12, k) > 0$ for all $k > 13$, moreover $g(5, 12, 51) = 9$.

5. Other Sprague–Grundy values

Theorem 2 can be extended to other Sprague–Grundy values besides 0. It also generalizes Nivasch's result regarding 2-pile Euclid [9]. We shall use $g(a, b, c)$ to denote the Sprague–Grundy function of the position (a, b, c), omitting the second set of parentheses. We now show that there are exactly a x-values for which $g(a, b, x) = s$, for all nonnegative integers s.

Theorem 4. *In the game of 3-Euclid, for every nonnegative integer $s \geqslant 0$, $g(a, b, x) = s$ for exactly a values of x, where a and b are constants with $a \leqslant b$ and x is a variable with $x \geqslant 1$.*

Proof. Again, we show that there is exactly one such x-value in each residue class mod a. Just as in the special case $s = 0$, it is easily seen that there cannot be more than one value in the same residue class. Now, we show that there is at least one x-value for which $g(a, b, x) = s$, for $x \equiv m \bmod a$, for all $m : 0 \leqslant m < a$.

Suppose to the contrary that $g(a, b, x) \neq s$ for all $x \equiv m \bmod a$. Consider first the case $b > a$. As above, let $p = \lceil b/a \rceil - 1$, and choose k such that $k \equiv m \bmod a$ and $b \leqslant k < b + a$. Let t be the least integer such that $k + (p + s + t - 1)a - tb \leqslant 0$; such an integer is guaranteed to exist since $b > a$, and in fact, $t = \lceil (k + (p + s - 1)a)/(b - a) \rceil$. Consider the sequence of positions $(a, b, k), (a, b, k + a), (a, b, k + 2a), \ldots, (a, b, k + (p + s + t - 1)a)$. In this sequence of $p+s+t$ positions, s of the members may have Sprague–Grundy values between 0 and $s-1$. The remaining $p + t$ positions in the sequence have Sprague–Grundy values at least s, so it is possible to move to a position with Sprague–Grundy value s in each of them. Each of the p 1–2 moves, and the $t - 1$ 2–3 moves can lead to a position of Sprague–Grundy value s in only one position in the sequence, else it would be possible to move between two positions of the same Sprague–Grundy value. Hence, in at least one of the $p + s + t$ positions in the sequence, we can remove a multiple of a from x to reach a position (a, b, x') with $g(a, b, x') = s$. But this is a contradiction, since $x \equiv x' \equiv m \bmod a$.

In the case $a = b$, consider the position $(a, a, m + (s + 1)a)$. For $s \geqslant 1$, each of the s positions in the sequence $(a, a, m + a), (a, a, m + 2a), \ldots, (a, a, m + sa)$ has a Sprague–Grundy value of at most s, the maximum number of legal moves, and none has the same value. If any has value s, we are done. Otherwise, they take the values $0, 1, \ldots, s-1$, and we see that $g(a, a, m + (s + 1)a) = s$. The proof is complete. □

We note that an upper bound on x so that $g(a, b, x) = s$ can be easily established similarly to Theorem 3.

6. On periodicity and the misère version

In this section we briefly mention a result concerning the periodicity of the P position and some results for the misère version.

The following theorem guarantees the periodicity of the P positions in b and c for any fixed a. It will be proven in a subsequent paper by the first author, Collins [2].

Theorem 5. *If $b > a^2$ and $c > a^2$, then $(a, b, c) \in P$ if and only if $(a, b + a, c + a) \in P$.*

This theorem makes finding the N/P value of large positions more efficient. Although this theorem gives us a periodicity of positions with Sprague–Grundy value 0, we know little about the periodicity of the Sprague–Grundy function, in general. One of the referees noted that for nonzero Sprague–Grundy values s, computer experiments seem to indicate that there is also periodicity analogous to Theorem 5, but the period is not a itself but a multiple of a, increasing as s increases. We also note that it is easy to prove that the sequence $g(2, i, i + 1), i \geqslant 1$, is purely periodic with period (0,1), and the sequence $g(3, i, i + 2), i \geqslant 1$, seems to become periodic with period (0, 1, 0) (cf. Fig. 1). In both cases, the positions of the zeros conform to the pattern given in Theorem 5.

In the misère version the player with no move wins. Let P_m and N_m be the P and N positions for the misère version, respectively. Clearly, $(a, a, a) \in \mathrm{N}_m$. As we mentioned in Section 2, for $a = 1$ the game with initial position $(1, b, c)$ is equivalent to misère Nim with pile sizes of $b - 1$ and $c - 1$. For example, $(1, 2, 2) \in \mathrm{N}_m$. For $a = 2$ and $b \neq 2, 4$, one can prove that $(a, b, b) \in \mathrm{P}_m$, by induction on b.

We can generalize Theorem 1 and get that $(a, b, b) \in \mathrm{P}_m$ for $b \neq a, 2a$. Otherwise, Theorems 2–4 and their proofs also generalize to the misère version; however, in this case, periodicity of the P positions remains an open question.

In conclusion we add that there is a significant difference in the techniques applied and results obtained between the original 2-pile Euclid and 3-Euclid. In the former one the P positions, and in fact, the sets of positions with any given

Sprague–Grundy value s exhibit a highly regular and geometrically appealing structure and can be easily characterized [9]. In contrast, the P-positions of 3-Euclid behave somewhat chaotically, and we were able to present only their partial characterization.

Acknowledgments

We wish to thank the referees for making constructive suggestions and comments that improved the presentation of this paper.

References

[1] A.J. Cole, A.J.T. Davie, A game based on the Euclidean algorithm and a winning strategy for it, Math. Gaz. 53 (1969) 354–357.
[2] D. Collins, The periodicity theorem, 2005, in preparation.
[3] D. Collins, Variations on a theme of Euclid, Integers 5 #G03 (2005) 1–12.
[4] A. Fraenkel, Euclid and Wythoff games, Discrete Math. 304 (2005) 65–68.
[5] J.W. Grossman, A nim-type game, problem #153, Math. Magazine 70 (1997) 382.
[6] V. Gurvich, On the misere version of game Euclid and miserable games, Discrete Math. 307 (2007) 1199–1204.
[7] T. Lengyel, A nim-type game and continued fractions, The Fibonacci Quart. 41 (2003) 310–320.
[8] T. Lengyel, On calculating the Sprague–Grundy function for the game Euclid, in: The 11th International Conference on Fibonacci Numbers and Their Applications, 2004, Kluwer, Dordrecht, pp. 169–175.
[9] G. Nivasch, The Sprague–Grundy function of the game Euclid, Discrete Math. 306 (21) (2006) 2798–2800.
[10] E.I. Spitznagel, Properties of a game based on Euclid's algorithm, Math. Magazine 46 (1973) 87–92.
[11] P.D. Straffin, A nim-type game, solution to problem #153, Math. Magazine 71 (1998) 194–195.