ARITHMETIC PROPERTIES OF LACUNARY SUMS OF BINOMIAL COEFFICIENTS

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Abstract

We study lacunary sums of binomial coefficients from the point of view of some arithmetic properties of these sums. We develop new recurrence relations and analyze some p-adic properties for a prime p. The main tools are generating functions, recurrence relations and some congruential and divisibility properties of the binomial coefficients.

1. Introduction

We study certain lacunary sums of binomial coefficients. We assume that n, m, i, and $j, j < m$ are nonnegative integers. We define the sequence $\{U_n\}_{n\geq 0}$

$$U_n = U(n; m, j, i) = \binom{mn+j}{i}$$

and the corresponding generating function

$$l(x) = l(x; m, j, i) = \sum_{k=0}^{\infty} U_k x^k = \sum_{k=0}^{\infty} \binom{mk+j}{i} x^k,$$

and consider the sum

$$T_n = T(n; m, j, i) = \sum_{k=0}^{n} \binom{mk+j}{i}$$

and the generating function of the sequence $\{T_n\}_{n\geq 0}$

$$f(x) = f(x; m, j, i) = \sum_{n=0}^{\infty} T(n; m, j, i) x^n.$$

To simplify notation, we drop the parameters and use the short forms T_n, U_n, $f(x)$, and $l(x)$ whenever this convention does not cause confusion.

We have found little in the literature on the properties of T_n, although Takács (cf. [7]) analyzed some of its properties and derived the explicit expression $T_n = T(n; m, j, i) = P(m(n+1)+j, i, m) - P(j, i, m)$ with

$$P(x, i, m) = \sum_{s=0}^{i} \binom{x/m}{s+1} \sum_{t=0}^{s} (-1)^{s-t} \binom{s}{t} \binom{mt}{i},$$

where we define $\binom{x}{0} = 1$ for any x and $\binom{x}{s} = x(x-1)\dots(x-s+1)/s!$ for any x and $s = 1, 2, \dots$. The computational advantage of this formula is that the second sum does not depend on x (and thus, on j after substitution). We take a different approach that emphasizes the arithmetic properties of the sequence $\{T_n\}_{n\geq 0}$. We focus on establishing recurrence relations satisfied by the underlying quantities and applying some congruences. We use generating functions, recurrence relations and some congruential and divisibility properties of the binomial coefficients. Some applications of $\{T_n\}_{n\geq 0}$ are considered in [6] and some problems implicitly concern the quantity $T(n; 2, j, 2)$ with $j = 0$ and 1, cf. [2] for example.

The main results are presented in Lemma 1 and Theorems 1–3, 6–11, and Corollary 1. We include the proofs and some examples (cf. Examples 1–5 in Sections 3–5). The next two sections include the results that yield the recurrence relations and their proofs. We present the p-adic results and their proofs in Sections 4 and 5. Section 6 is devoted to some applications.

We note that sometimes, people mean a different sum when referring to a lacunary sum of binomial coefficients. The p-adic and congruential study of these sums is also a popular topic, cf. [4], [8], and [9].

2. The Main Recurrence Relations

In this section we state the main results leading to recurrence relations in Lemma 1 and Theorems 1 and 2.

Lemma 1. *In the case of $m = 1$ and $j = 0$ we have that*

$$T_n = T(n; 1, 0, i) = \binom{n+1}{i+1} \tag{1}$$

and

$$f(x; 1, 0, i) = x^i/(1-x)^{i+2}. \tag{2}$$

Thus, we obtain the following homogeneous linear recurrence for T_n:

$$\sum_{s=0}^{i+2} \binom{i+2}{s} (-1)^s T_{n-s} = 0, n \geq i+2. \tag{3}$$

Theorem 1. *Let $n, m, i,$ and $j, j < m$ be nonnegative integers. We have that*

(1) $f(x; m, j, i) = \frac{p(x)}{(1-x)^{i+2}}$ *where $p(x) = p(x; m, j, i)$ is a polynomial of degree i with integer coefficients and the homogeneous linear recurrence* (3) *holds for $T_n = T(n; m, j, i)$ with $n \geq i+2$*

(2) $T_n = T(n; m, j, i)$ *is a polynomial of degree $i+1$ in variable n with rational coefficients*

(3) the leading coefficient of the polynomial for T_n is $m^i/(i+1)!$.

Note: If we refer to a polynomial and leave out the reference to its variable then the first argument on the parameter list or the subscript is supposed to be the variable of the polynomial.

Remark 1. Surprisingly, the recurrence relation (3) does not depend on j and m.

We also explore the relation between $l(x)$ and $f(x)$. Clearly, an alternative derivation of the generating function of the sequence $\{T_n\}_{n\geq 0}$ is given by

$$\frac{l(x)}{1-x} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{mk+j}{i} x^n = \sum_{n=0}^{\infty} T_n x^n = f(x). \tag{4}$$

Theorem 2. *Let $n, m, i,$ and $j, j < m$ be nonnegative integers. The homogeneous linear recurrence*

$$\sum_{s=0}^{i+1} \binom{i+1}{s} (-1)^s U_{n-s} = 0, n \geq i+1 \tag{5}$$

holds for $U_n = U(n; m, j, i)$ with $n \geq i+1$. We have that

$$l(x; m, j, i) = \frac{r(x)}{(1-x)^{i+1}} \tag{6}$$

where $r(x) = r(x; m, j, i)$ is a polynomial of degree i with integer coefficients.

3. Proofs for the Recurrence Relations

In this section we use generating functions to derive and prove recurrence relations that are interesting on their own and will be used in Section 4.

Proof of Lemma 1. Identities (1) and (2) are obvious by the so called hockey-stick identity

$$\binom{i}{i} + \binom{i+1}{i} + \cdots + \binom{n}{i} = \binom{n+1}{i+1} \tag{7}$$

and

$$\begin{aligned} f(x;1,0,i) = \sum_{n=0}^{\infty} T_n x^n &= \sum_{n=i}^{\infty} \binom{n+1}{i+1} x^n = x^i \sum_{n=0}^{\infty} \binom{n+i+1}{i+1} x^n \\ &= x^i \sum_{n=0}^{\infty} \binom{-(i+2)}{n} (-x)^n = x^i/(1-x)^{i+2}. \end{aligned} \tag{8}$$

The denominator of (2) guarantees (3). □

We prove Theorem 2 before Theorem 1.

Proof of Theorem 2. Now we show that $l(x) = r(x)/(1-x)^{i+1}$ where $r(x) = r(x;m,j,i)$ is a polynomial of degree i with integer coefficients. For example, with $m = 1$ and $j = 0$ we get that $l(x) = x^i/(1-x)^{i+1}$ in a similar fashion to (8).

For all $m \geq 1$, we prove by induction on $i \geq 0$ that the denominator of $l(x) = l(x;m,j,i)$ in (6) is $(1-x)^{i+1}$ independently of j while the numerator is a polynomial of degree i with integer coefficients. The statement is obviously true for $i = 0$ since $l(x;m,j,0) = 1/(1-x)$. We assume that it is also true for every integer $0, 1, 2 \ldots, i-1$ and prove it for i, too. To this end, we observe that

$$\begin{aligned} \binom{mk+j}{i} - \binom{m(k-1)+j}{i} = &\binom{m(k-1)+m+j-1}{i-1} \\ &+ \binom{m(k-1)+m+j-2}{i-1} + \ldots \\ &+ \binom{m(k-1)+j}{i-1} \end{aligned}$$

by the hockey-stick identity. After multiplying both sides by x^k and summing up the terms we obtain

$$\begin{aligned}(1-x)l(x;m,j,i) &= \sum_{k=1}^{\infty}\left(\binom{mk+j}{i}-\binom{m(k-1)+j}{i}\right)x^k\\ &= \sum_{s=0}^{j-1} l(x;m,s,i-1) + x\sum_{s=j}^{m-1} l(x;m,s,i-1)\end{aligned}$$

for $j \geq 1$ and $(1-x)l(x;m,0,i) = x\sum_{s=0}^{m-1} l(x;m,s,i-1)$ for $j=0$. The result follows after both sides are divided by $1-x$. The degree of the polynomial in the numerator also follows.

The form of the generating function $l(x)$ guarantees the validity of the recurrence (5). $\square$

Proof of Theorem 1. To prove part (1) we simply use Theorem 2 and identity (4). The form of the generating function $f(x)$ guarantees the validity of the recurrence (3).

Now we turn to parts (2) and (3) of the statement. We use a bivariate generating function $g(x,y)$ in which y marks the summation variable k.

$$\begin{aligned}g(x,y) &= \sum_{i=0}^{mn+j}\sum_{k=0}^{n}\binom{mk+j}{i}x^i y^k = \sum_{k=0}^{n} y^k \sum_{i=0}^{mk+j}\binom{mk+j}{i}x^i\\ &= \sum_{k=0}^{n} y^k(1+x)^{mk+j} = (1+x)^j\frac{1-(y(1+x)^m)^{n+1}}{1-y(1+x)^m}.\end{aligned}$$

In fact, with $y=1$ we derive the generating function

$$\begin{aligned}h(x) = g(x,1) &= \sum_{i=0}^{mn+j} T(n;m,j,i)x^i = (1+x)^j\frac{(1+x)^{m(n+1)}-1}{(1+x)^m-1}\\ &= \frac{(1+x)^{m(n+1)+j}-(1+x)^j}{(1+x)^m-1}.\end{aligned}$$

Since $(1+x)^m - 1 = mx\left(1+\frac{m-1}{2}x+\frac{(m-1)(m-2)}{6}x^2+\dots\right)$, it follows that

$$\begin{aligned}T(n;m,j,i) &= [x^i]\ h(x)\\ &= \sum_{s=0}^{i+1}\frac{1}{m}\left(\binom{m(n+1)+j}{s}-\binom{j}{s}\right)\times \qquad (9)\\ &\quad\times [x^{i+1-s}]\frac{1}{1+\frac{m-1}{2}x+\frac{(m-1)(m-2)}{6}x^2+\dots}\end{aligned}$$

where the denominator in the last expression does not depend on n. Therefore, we conclude that T_n is a polynomial in n of degree $i+1$ with rational coefficients and the leading coefficient, i.e., that of n^{i+1}, is $m^i/(i+1)!$. □

Note that Takács also proved parts (2) and (3) of Theorem 1 in [7].

Remark 2. We note that identities (4) and (6) also yield that T_n is a polynomial in n of degree $i+1$.

Remark 3. The fact (3) with (9) guarantees the asymptotic result

$$T(n;m,j,i) = \sum_{k=0}^{n} \binom{mk+j}{i} \sim \frac{1}{m}\binom{m(n+1)}{i+1}$$

as $n \to \infty$.

Example 1. For $m = i = 2$ we can easily derive that $T(n;2,0,2) = \frac{1}{6}(4n^3+3n^2-n) = \frac{1}{6}n(n+1)(4n-1)$ in agreement with the initial segment 0, 1, 7, 22, 50, 95, 161, 252 of the sequence. In a similar fashion, we get $T(n;2,1,2) = \frac{1}{6}(4n^3+9n^2+5n) = \frac{1}{6}n(n+1)(4n+5)$ in agreement with the initial segment 0, 3, 13, 34, 70, 125, 203, 308 of the sequence. We return to this example in Section 6.

4. P-adic Properties

The sequence $\{T(n;m,j,i)\}_{n\geq 0}$ is periodic mod a for any integer $a \geq 1$ by (1) of Theorem 1. Kwong analyzed the question of finding the minimum period of similar recurrent sequences in [5]. Here we are interested in the p-adic properties of the sequence for a prime p.

For an integer a, the p-adic order $\nu_p(a)$ of a is the highest power of prime p which divides a. We set $\nu_p(0) = \infty$ and $\nu_p(a/b) = \nu_p(a) - \nu_p(b)$ if both a and b are integers.

Let $a(x)$ and $b(x)$ be polynomials in variable x. We say that $b(x)$ divides $a(x)$ and write that $a(x) \equiv 0 \bmod b(x)$ if there exists a polynomial $c(x)$ so that $a(x) = c(x)b(x)$. For instance, when we say that n or $n+1$ divides $T(n;m,j,i)$ we mean the divisibility of the corresponding polynomials.

The analysis of the p-adic order of $\nu_p(T(n;p,j,i))$ amounts to the p-adic evaluation of a polynomial of degree $i+1$ with rational coefficients according to fact (2) of Theorem 1. Factoring T_n facilitates this job. It seems that many factors are linear or quadratic functions over the rationals. After factorization we obtain the p-adic order for any particular set of parameters. However, it is desirable to discover some general features of the sequence that can be used without the analysis of the particular cases. We derive Theorems 3, 6–10, and Corollary 1 to shed light on the behavior of $\nu_p(T_n)$. Examples 2–4 illustrate some of the general features.

Theorem 3. *We have that* $T(n;p,0,i) \equiv 0 \bmod p$ *if* $i \not\equiv 0 \bmod p$.

Proof. The statement follows since for all terms in the definition of T_n we have that $\binom{kp}{i} \equiv 0 \bmod p$ if $i \not\equiv 0 \bmod p$ by carry counting according to Kummer's theorem for the p-adic order of binomial coefficients [3, p.79]. $\square$

The case with $i \equiv 0 \bmod p$ is discussed in Theorem 6, and its proof requires some preparation. In fact, we need Theorems 4 and 5.

Theorem 4 (Lucas, 1877)**.** *Let* $N = n_0 + n_1p + \cdots + n_dp^d$ *and* $M = m_0 + m_1p + \cdots + m_dp^d$, *with* $0 \le n_i, m_i \le p-1$ *for each* i, *be the base* p *representations of* N *and* M, *respectively. For a prime* p *we have*

$$\binom{N}{M} \equiv \binom{n_0}{m_0}\binom{n_1}{m_1}\cdots\binom{n_d}{m_d} \bmod p.$$

Theorem 5 (Jacobstahl–Kazandzidis congruence, cf. Corollary 11.6.22 [1])**.** *Let* M *and* N *be such that* $0 \le M \le N$ *and* p *prime. We have*

$$\binom{pN}{pM} \equiv \begin{cases} \left(1 - \frac{B_{p-3}}{3}p^3NM(N-M)\right)\binom{N}{M} & \bmod p^4NM(N-M)\binom{N}{M}, \ \text{if } p \ge 5, \\ (1+45NM(N-M))\binom{N}{M} & \bmod p^4NM(N-M)\binom{N}{M}, \ \text{if } p = 3, \\ (-1)^{M(N-M)}P(N,M)\binom{N}{M} & \bmod p^4NM(N-M)\binom{N}{M}, \ \text{if } p = 2, \end{cases}$$

where $P(N,M) = 1+6NM(N-M)-4NM(N-M)(N^2-NM+M^2)+2(NM(N-M))^2$ *and* B_n *stands for the* n*th Bernoulli number.*

Theorem 6. *For* $0 \le j < p$ *and* $t \ge 0$ *integers we have that*

$$T_n = T(n;p,j,tp) \equiv \binom{n+1}{t+1} \bmod p.$$

In addition, we also have that

$$T_n = T(n;p,0,tp) \equiv \binom{n+1}{t+1} \bmod p^q$$

with

$$q = \begin{cases} 2, & \text{if } p = 2 \text{ or } 3, \\ 3, & \text{if } p \ge 5. \end{cases}$$

Furthermore, if $p = 2$ *and* $t \ge 0$ *is even then*

$$T_n = T(n;2,0,2t) \equiv \binom{n+1}{t+1} \bmod 2^{\nu_2(t)+1}.$$

Proof of Theorem 6. The result follows by Lucas' theorem and the Jacobstahl–Kazandzidis congruence, respectively. Indeed, for $0 \le j < p$ we get that $\binom{kp+j}{tp} \equiv \binom{k}{t}\binom{j}{0} \equiv \binom{k}{t} \bmod p$ by Theorem 4 and $T_n = T(n;p,j,tp) \equiv \sum_{k=0}^{n}\binom{k}{t} = \binom{n+1}{t+1} \bmod p$ by identity (7).

On the other hand, for $p \ge 3$ we have $\binom{kp}{tp} \equiv \binom{k}{t} \bmod p^q$ by Theorem 5, and it follows that $T_n = T(n;p,0,tp) \equiv \sum_{k=0}^{n}\binom{k}{t} = \binom{n+1}{t+1} \bmod p^q$. If $p=2$ and t is even then we can use the 2-exponent $\nu_2(t)+1$ by the same reasoning and utilizing the factor t in the modulus. If $p=2$ and t is odd then the terms of T_n satisfy the congruence

$$\binom{2k}{2t} \equiv (-1)^{t(k-t)}\binom{k}{t} \bmod 4,$$

which is $\binom{k}{t}$ if k is odd and $-\binom{k}{t} \equiv 0$ or $2 \bmod 4$ and thus, $-\binom{k}{t} \equiv \binom{k}{t} \bmod 4$ if k is even. The statement follows. □

Now we turn to the question about the magnitude of $\nu_p(T_n)$.

Theorem 7. *For $i \ge 0$ we have that $n+1$ divides $T(n;m,j,i)$.*

Proof. The case with $i=0$ is immediate, and otherwise, the statement follows by (9) since

$$\binom{m(n+1)+j}{s} - \binom{j}{s} \equiv 0 \bmod (n+1). \tag{10}$$

In fact, if $i<j$ then for any $s', 0 \le s' \le s \le i+1 \le j$ we have $m(n+1)+j-s' \equiv j-s' \bmod (n+1)$ which yields the congruence (10).

If $i \ge j$ and $s \le j$ then the above argument applies. If $j < s \le i+1$ then

$$\binom{m(n+1)+j}{s} \equiv \frac{j(j-1)\dots(j-s+1)}{s!} \equiv 0 \equiv \binom{j}{s} \bmod (n+1), \tag{11}$$

and (10) follows again. □

By setting $n = p^N - 1$ with a large N, Theorem 7 implies

Corollary 1. *The p-adic order $\nu_p(T(n;m,j,i))$ can be arbitrarily large for any prime p.*

Numerical experiments suggest the following improvement. Although, at first, it seems difficult to prove that

$$\sum_{k=0}^{n}\binom{mk+j}{i}$$

is divisible by n, the previous theorem comes to the rescue.

Theorem 8. *For $0 \le j < i$ we have that $n(n+1)$ divides $T(n;m,j,i)$.*

Proof. Theorem 7 takes care of divisibility by $n+1$. For the other part, we divide the sum into two parts

$$T(n;m,j,i) = T(n-1;m,j,i) + \binom{mn+j}{i}.$$

The first term is divisible by n by Theorem 7. As far as the second term is concerned, we proceed in a similar fashion to congruence (11)

$$\binom{mn+j}{i} \equiv \frac{j(j-1)\dots(j-i+1)}{i!} \equiv 0 \bmod n \tag{12}$$

since $0 \le j < i$, and the proof is complete. $\square$

Remark 4. As an interesting observation we note that, for $n = p$ prime with $(m,p) = 1$, the lists $\{\binom{mk+j}{i} \bmod p\}_{k=0}^{n}$ of remainders are identical for $j = 1, 2, \dots i-1$. We can get one list from another by applying circular shifts as we move from $j = j_1$ to $j = j_2$ given $0 \le j_1 < j_2 < i$ with the exception of the first element in the list that belongs to j_1 and the last element of the list corresponding to j_2. Both these elements are equal to 0 as in (12). (If $n = p$, $(m,p) > 1$ and $0 \le j < i < m$ then the lists consist of zeros only.) This guarantees that $T(p;m,j,i) \bmod p$ is the same for all $j : 0 \le j \le i-1$. The general idea is that for all $k : 0 \le k \le n$ with some shift C we have that

$$\binom{m(k+C)+j_1}{i} \equiv \binom{mk+j_2}{i} \bmod p,$$

since for all $s : 0 \le s \le i$ the congruence $m(k+C)+j_1-s \equiv mk+j_2-s \bmod p$ is equivalent to $mC \equiv j_2 - j_1 \bmod p$; thus, $C \equiv m^{-1}(j_2-j_1) \bmod p$ can serve as the shift.

It is possible, however, that a subsequence of $\{\nu_p(T_n)\}_{n\ge 0}$ remains constant. A case in point is parts (a) and (b) of Theorem 9.

Theorem 9. *Let p be a prime. We set $n_0 = n_0(m,j,i) = \max\{0, \lceil \frac{i-j}{m} \rceil\}$, $T_s = T(s;m,j,i)$ and*

$$\underset{n_0 \le s \le n_0+i+1}{\operatorname{argmin}} \nu_p(T(s;m,j,i)) = s' \tag{13}$$

with $M = \nu_p(T_{s'})$.

(a) If $i = p-2$ then there is an arithmetic sequence $\{s' + Kp\}_{K\ge 0}$ with some integer $s' \ge n_0$ so that the subsequence $\{\nu_p(T_{s'+Kp})\}_{K\ge 0}$ of the original sequence $\{\nu_p(T_n)\}_{n\ge 0}$ remains constant. We set s' by (13) and have, with $T_{s'+Kp} = T(s' + Kp;m,j,p-2)$, that

$$\nu_p(T_{s'+Kp}) = M \tag{14}$$

for all nonnegative integers K.

(b) If $i \geq 0$ and the minimum in (13) *is uniquely taken at s', i.e., $M = \nu_p(T_{s'}) < \nu_p(T_s)$ for all $s : s \neq s', n_0 \leq s \leq n_0 + i + 1$, then $\nu_p(T_{s'+K(i+2)}) = M$ holds with $T_{s'+K(i+2)} = T(s' + K(i + 2); m, j, i)$.*

(c) In general, we have the universal lower bound $M \leq \nu_p(T_n)$ for all $n \geq s'$.

Proof. For part (a) we assume that p is a prime and $i = p - 2$. In this case $\binom{i+2}{s} \equiv 0 \bmod p$ for $1 \leq s \leq i+1$ and the recurrence (3) taken mod p guarantees that $M = \nu_p(T_{s'}) = \nu_p(T_{s'+i+2}) = \nu_p(T_{s'+2(i+2)}) = \ldots$. For part (b) the mod p^{M+1} version of recurrence (3) yields the result. Part (c) follows by mathematical induction on $n \geq n_0(m, j, i)$ and (3). □

Example 2. We can easily see that for $n \geq 1$

$$\begin{aligned}\nu_5(T(n; 5, 2, 3)) &= \nu_5\left(\frac{5}{24}n(n+1)(5n+2)(5n+7)\right)\\ &= \begin{cases}\nu_5(n) + 1, & \text{if } n \equiv 0 \bmod 5,\\ \nu_5(n+1) + 1, & \text{if } n \equiv 4 \bmod 5,\\ 1, & \text{otherwise.}\end{cases}\end{aligned}$$

The minimum mod 5 and mod 25 periods are $\pi(5) = 1$ and $\pi(5^2) = 5$, and we have the lists $\{T_n\}_{n\geq 0} = \{0, 35, 255, 935, 2475, 5400, \ldots\}$ and $\{\nu_5(T_n)\}_{n=1}^{n=5} = \{1, 1, 1, 2, 2\}$. We conclude that $\nu_5(T_{5^n-1}) = \nu_5(T_{5^n}) = n + 1$ if $n \geq 1$ and $\nu_5(T_{s'+5n}) = 1$ if $n \geq 0$ and $s' = 1, 2, 3$ by Theorem 9.

Remark 5. We note that part (b) of Theorem 9 does not seem to have practical value as we found only a few sets of parameters when its conditions are satisfied: Example 4 (and the same setting with other values of m) and the trivial cases of $(m, j, i) = (2, 0, 0)$ and $(2, 1, 0)$ with $p = 2$. Note that these cases are also covered by part (a). On the other hand, sometimes it is easy to see why the p-adic order remains constant without invoking Theorem 9. Numerical experiments suggest that for small values of i we often encounter the case where $\nu_p(T_n) = A\nu_p(n)+B\nu_p(n+1)+C$ with some positive rational constants A, B, and C. In this case, $\nu_p(T_n) = C$ provided that $n \not\equiv 0$ or $-1 \bmod p$ as the next two examples highlight (with $p = 3$ and 5), and Example 2 also fits this pattern.

Example 3. We obtain that for $n \geq 1$

$$\nu_5(T(n; 3, 1, 2)) = \nu_5\left(\frac{3}{2}n(n+1)^2\right) = \begin{cases}\nu_5(n), & \text{if } n \equiv 0 \bmod 5,\\ 0, & \text{if } n \equiv 1, 2, 3 \bmod 5,\\ 2\nu_5(n+1), & \text{if } n \equiv 4 \bmod 5.\end{cases}$$

The minimum mod 5 and mod 25 periods are $\pi(5) = 5$ and $\pi(5^2) = 25$, and we have the lists $\{T_n\}_{n\geq 0} = \{0, 6, 27, 72, 150, 270, 441, 672, 972, 1350, \dots\}$ and $\{\nu_5(T_n)\}_{n\geq 1} = \{0, 0, 0, 2, 1, 0, 0, 0, 2, 1, ..., 0, 0, 0, 4, 2, 0, 0, 0, 2, 1, ...\}$. In particular, $\nu_5(T_{5n+a}) = 0$ with $a = 1, 2, 3$ if $n \geq 0$, although (except for part (c)) Theorem 9 does not apply here.

Example 4. We set $p = 3, m = 3, j = 0$, and $i = 1$. In this example, both parts (a) and (b) of Theorem 9 apply, yet the shortcut presented in Remark 5 immediately guarantees that $\nu_3(T_{3n+1}) = 1$ for all $n \geq 0$ since $T_n = T(n; 3, 0, 1) = \frac{3}{2}n(n+1)$. Here the minimum mod 3 and mod 9 periods are $\pi(3) = 1$ and $\pi(3^2) = 3$, and we have the lists $\{T_n\}_{n\geq 0} = \{0, 3, 9, 18, 30, 45, 63, 84, 108, 135 \dots\}$ and $\{\nu_3(T_n)\}_{n\geq 1} = \{1, 2, 2, 1, 2, 2, 1, 3, 3, \dots\}$.

We obtain the p-sected modular version of recurrence (3) in the following theorem. We note that p-sections might help in uncovering some inconspicuous arithmetic properties.

Theorem 10. *With $n \geq (t+1)p$ and $t \geq 0$, we have for $T_n = T(n; p, 0, (t+1)p-2)$ that*

$$\sum_{s=0}^{t+1} \binom{(t+1)p}{sp} (-1)^{sp} T_{n-sp} \equiv 0 \bmod p. \tag{15}$$

Remark 6. Note that typically the exponent of p can be increased because of part (c) of Theorem 9.

Proof of Theorem 10. We apply recurrence (3). Since now $i + 2 = (t+1)p$, we get that

$$\binom{(t+1)p}{l} \equiv 0 \bmod p$$

if $l \not\equiv 0 \bmod p$ by Theorem 4. The remaining terms yield (15). $\square$

5. The Rate of p-adic Convergence for Some Subsequences

We investigate the rate of p-adic convergence of the subsequence $\{T(ap^n + b; m, j, i)\}_{n\geq 0}$ with $a \geq 1$ and b integers. We rely on the fact the T_n is a polynomial in n. The proof of the following theorem is straightforward.

Theorem 11. *Let $P(n) = \sum_{i=0}^{k} a_i n^i$ be a polynomial in n with rational coefficients $a_i, 0 \leq i \leq k$. Let p be a prime, $a \geq 1$ and b integers so that $(a,p) = 1$ and $ap^{n-1} + b > 0$. We set $j = \min\{i | i \geq 1$ and $a_i \neq 0\}$ and $l = \nu_p(a_j)$, then*

$$\nu_p(P(ap^n + b) - P(ap^{n-1} + b)) = \begin{cases} j(n-1) + l, & if\, b = 0, \\ n - 1 + \nu_p(P'(b)), & if\, b \neq 0, \end{cases}$$

for any sufficiently large integer n.

Example 5. For $P(n) = 3n^5+2n^4+3n^3+1$ we get that $\nu_2(P(3\times 2^n)-P(3\times 2^{n-1})) = 3(n-1)$ for $n \geq 2$, while $\nu_2(P(3 \times 2^n + 1) - P(3 \times 2^{n-1} + 1)) = n + 4$ for $n \geq 7$ since $j = 3$, $l = 0$, and $P'(1) = 32$.

Theorem 11 gives the rate of p-adic convergence of $T_{ap^n+b} = T(ap^n + b; m, j, i)$ as $n \to \infty$. Although, typically $\nu_p(T_{ap^n+b} - T_{ap^{n-1}+b})$ increases by one as $n \to \infty$ it is not necessarily the case. For instance, if $T(n; m, j, i)$ is divisible by n^2 but not by n^3, then $\nu_p(T(ap^n; m, j, i) - T(ap^{n-1}; m, j, i))$ increases by two as $n \to \infty$. For example, this is the case for $T(n; 2, 0, 3)$, $T(n; 2, 1, 5)$, $T(n; 3, 0, 2)$, and $T(n; 4, 0, 5)$.

Remark 7. Theorem 11 can be easily extended to p-adic power series in n.

6. Applications of $\{T_n\}_{n\geq 0}$

To illustrate the interest in the sequence $\{T_n\}_{n\geq 0}$ and for some background information of its use, we include two applications. In [6] the following asymptotic result was proven.

Theorem 12 (Theorem 3, [6])**.** *Let $i, m,$ and n be positive integers and j be an integer so that $0 \leq j < \min\{i + 1, m\}$. We have that*

$$\sum_{k=0}^{n} \binom{mk+j}{i} \sim \frac{1}{m}\binom{m(n+1)+j}{i+1}\left(1 - \frac{i+1}{2n}\left(1 - \frac{1}{m}\right)\right)$$

for $1 \leq i < n^{1/4}$ as $n \to \infty$.

This result with $j = 0$ was the main tool to answer a question regarding card ranks in an asymptotic sense. We pick the first i cards from the top of a standard deck and check the ranks of these cards as they show up. The kings have the highest rank, 13, while the aces have the lowest, 1. In the generalized case the deck of cards has n denominations (of face values $1,\ 2,\ \ldots,\ n$) in each of m suits; thus, $n = 13$ and $m = 4$ for the standard deck. For the probability p_i that the first card has the uniquely highest rank among the top i cards of the card deck we obtain

Theorem 13 (Theorem 1, [6])**.** *Let $i, m,$ and n be positive integers and p_i as defined above. Then*

$$p_i \approx \frac{1}{i} - \frac{1}{2n}\left(1 - \frac{1}{m}\right), \tag{16}$$

for $2 \leq i \leq n^{1/4}$ as $n \to \infty$.

Another application concerns counting triangles in triangles, e.g., [2]. The total number of triangles in an equilateral triangle of side n tiled by equilateral triangles of side one is $\frac{1}{16}\left(4n^3+10n^2+4n-1+(-1)^n\right)$ which can be confirmed by observing that

$$T(n+1;1,0,2)+\begin{cases} T(\lfloor n/2\rfloor;2,0,2), & \text{if } n \text{ is even,}\\ T(\lfloor n/2\rfloor;2,1,2), & \text{if } n \text{ is odd}\end{cases}$$

counts the included triangles. Indeed, if the base triangle is pointing up then the first and second terms count the triangles pointing up and down, respectively. The second term is calculated explicitly in Example 1.

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