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STIRLING NUMBERS OF THE SECOND KIND, ASSOCIATED STIRLING NUMBERS OF THE SECOND KIND AND NEWTON–EULER SEQUENCES

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Abstract

We show that the sequence $\{k!S(n,k)\}_{n=1}^{\infty}$ formed from the Stirling Numbers of the Second Kind $S(n,k)$ is a Newton–Euler sequence for all $k \geq 1$. This property guarantees some alternative proofs of congruential properties. Associated Stirling numbers do not form Newton–Euler Sequences, thus, we use other techniques to obtain some divisibility properties. We also derive a new ordinary generating function of the associated Stirling numbers $\{S_r(n,k)\}_{n=1}^{\infty}$ that relies on a recurrence in k.

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1. Introduction

The Stirling Number of the Second Kind $S(n,k)$ is defined as the number of ways to partition a set of n things into k nonempty subsets while the r-associated Stirling Number of the Second Kind $S_r(n,k), r \geq 1$, is a similar count with the extra stipulation that each of the k subsets has at least r elements.

To derive some congruential properties of these Stirling numbers we prove in Theorem 2 that the sequence $\{k!S(n,k)\}_{n=1}^{\infty}$ forms a Newton–Euler Sequence, cf. [9] and [3].

First we introduce the notion of Newton–Euler Pairs and Newton–Euler Sequences. Section 1 is devoted to some basic facts regarding Newton–Euler Pairs and contains one of the main results Theorem 2 while Section 2 has the proofs. Section 3 deals with r-associated Stirling Numbers of the Second Kind. Subsections 3.1 and 3.2 are devoted to the cases with $r = 2$ and $r > 2$, respectively, and some 2-adic divisibility properties are presented or conjectured on the underlying quantities and their differences (cf. Theorem 8, Corollary 1, and Conjectures 1 and 2). Some special cases with $k = 2,3,4$ if $r = 2$ and $k = 2,3$ if $r = 3$ are fully explored and explained, e.g., Remark 3. In Subsection 3.3 we derive a recurrence in k for the ordinary generating function $C_{r,k}(x) = \sum_{n=0}^{\infty} S_r(n,k)x^n = \sum_{n=kr}^{\infty} S_r(n,k)x^n$ of the associated Stirling Numbers of the Second Kind $\{S_r(n,k)\}_{n=1}^{\infty}$, cf. (3.21) and Theorem 10. Note that we could locate only the exponential generating function of these quantities in the literature.

The following definitions, statements, lemma and theorem are from [9].

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Definition 1. *If $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences satisfying $a_1 = b_1$ and $b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 = n a_n$ $(n > 1)$, then we say that (a_n, b_n) is a Newton–Euler Pair.*

We implicitly assume that $n \geq 1$ when referring to (a_n, b_n) as a Newton–Euler Pair.

Remark 1. *For each sequence $\{a_n\}_{n=1}^{\infty}$ there is a unique sequence $\{b_n\}_{n=1}^{\infty}$ such that (a_n, b_n) is a Newton–Euler Pair, and vice versa. If the sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ satisfy the relation $\sum_{k=0}^{n} a_k b_{n-k} = n a_n (n = 0, 1, 2, \ldots)$ and $a_0 \neq 0$ then $(a_n/a_0, b_n)$ is a Newton–Euler Pair.*

Definition 2. *If (a_n, b_n) is a Newton–Euler Pair and $a_n \in \mathbb{Z}$ for all $n = 1, 2, 3, \ldots$ then we say that $\{b_n\}$ is a Newton–Euler Sequence.*

Lemma 1. *Let $\{b_n\}$ be a Newton–Euler Sequence. Then clearly, $b_n \in \mathbb{Z}$ for all $n = 1, 2, 3, \ldots$.*

Theorem 1. *(Theorem by Du, Huang, and Li, cf. (1.1) in [9], also see [3]) For the $\{b_n\}$ Newton–Euler Sequence we have that*

$$\sum_{d|n} \mu(d) b_{\frac{n}{d}} \equiv 0 \pmod{n} \quad \text{and} \quad b_n \equiv b_{\frac{n}{p}} \pmod{p^t}$$

where μ is the Möbius function and p is a prime such that $p^t \mid n$.

One of our main results is

Theorem 2. *The sequence $\{k!S(n,k)\}_{n=1}^{\infty}$ with any fix $k \geq 1$ is a Newton–Euler Sequence.*

Thus Theorem 1 applies and we get

Theorem 3. *We have that*

$$\sum_{d|n} \mu(d) k! S\left(\frac{n}{d}, k\right) \equiv 0 \pmod{n}$$

and

$$k!S(n,k) \equiv k!S\left(\frac{n}{p}, k\right) \pmod{p^t}. \tag{1.1}$$

where μ is the Möbius function and p is a prime such that $p^t \mid n$.

Note that the first congruence in Theorem 3 was first derived in [7, Proposition 4.15] by a different approach. We also add that Theorem 3 can be used (cf. (6) in [6]) to obtain a partial proof of the general Theorem 4. Let k and n be positive integers, and let $d_2(k)$ and $\nu_2(k)$ denote the number of ones in the binary representation of k and the highest power of two dividing k, respectively.

Theorem 4. *(Theorem 2 in [6]) Let $c, k, n \in \mathbb{N}$ and $1 \leq k \leq 2^n$, then*

$$\nu_2\left(S(c2^n, k)\right) = d_2(k) - 1. \tag{1.2}$$

In order to prove Theorem 2 we need some preparations.

Theorem 5. *(Theorem 2.1 in [9]) Let $\{a_n\}$ and $\{b_n\}$ be two sequences. If $A(x) = 1 + \sum_{n=1}^{\infty} a_n x^n$ and $B(x) = \sum_{n=1}^{\infty} b_n x^n$ then the following statements are equivalent:*

(i) (a_n, b_n) *is a Newton–Euler Pair.*

(ii) $B(x) = xA'(x)/A(x)$.

(iii) $A(x) = e^{\int_0^x \frac{B(t)}{t}\,dt}$.

We will also need the rational expansion theorem for distinct roots.

Theorem 6. *(cf. p340, [4]) If $R(z) = P(z)/Q(z)$, where $Q(z) = q_0(1-\rho_1 z)\ldots(1-\rho_l z)$ and the numbers $(\rho_1, \ldots, \rho_l)$ are distinct, and if $P(z)$ is a polynomial of degree less than l, then*

$$R(z) = -\sum_{k=1}^{l} \frac{\rho_k P(1/\rho_k)}{Q'(1/\rho_k)} \frac{1}{1-\rho_k z}. \tag{1.3}$$

2. Proofs

Proof of Theorem 2. We set

$$B(x) = \sum_{n=k}^{\infty} k!S(n,k)x^n.$$

Note that

$$B(x) = \frac{k!x^k}{\prod_{j=1}^{k}(1-jx)} \tag{2.4}$$

according to the well-known “vertical” rational generating function [1, Theorem C, p207].

By Lemma 1, the case (ii) of Theorem 5, and Theorem 1 it is sufficient to prove that the generating function $A(x) = \sum_{j=0}^{\infty} a_j x^j$ has integral coefficients a_js. By Theorem 5 we know that

$$B(x) = xA'(x)/A(x) \tag{2.5}$$

implicitly provides us with the definition of $A(x)$. We derive by (2.4) and (2.5) that

$$\frac{k!x^{k-1}}{\prod_{j=1}^{k}(1-jx)} = (\ln A(x))'.$$

With $l = k, P(x) = k!x^{k-1}, Q(x) = \prod_{j=1}^{k}(1-jx)$, and $\rho_j = j$, and by Theorem 6 on partial fraction decomposition, it follows that

$$\frac{k!x^{k-1}}{\prod_{j=1}^{k}(1-jx)} = \sum_{j=1}^{k} \frac{\binom{k}{j}(-1)^{m(j)+1} j}{1-jx}$$

with $m(j) = j+1$ if k is even and $m(j) = j$ otherwise. Integrating on the right hand side leads to

$$\sum_{j=1}^{k} \binom{k}{j} (-1)^{m(j)} \ln(1-jx) = \ln\left(\prod_{j=1}^{n} (1-jx)^{(-1)^{m(j)}\binom{k}{j}}\right),$$

and thus, by $A(0)=1$, we conclude that

$$A(x)=\prod_{j=1}^{k}(1-jx)^{(-1)^{m(j)}\binom{k}{j}}.$$

This already guarantees that $a_n\in\mathbb{Z}$ for all $n=1,2,3,\ldots$ which by way of Lemma 1 yields that $b_n\in\mathbb{Z}$ for all $n=1,2,3,\ldots$. The proof is now complete. □

By the way, the following facts also hold.

Lemma 1. *For* $1\le j\le 2k-1$, *we have that*

$$a_j=\frac{k!S(j,k)}{j}$$

is an integer.

Proof. The proof can be derived from the proof of Theorem 2. It follows that the generating function $A(x)$ has integral coefficients, and it is easy to see that the coefficients with small indices are $a_0=1$ and $a_j=k!S(j,k)/j$ for $j=1,2,\ldots 2k-1$ since $b_j=k!S(j,k)=0$ for $j=0,1,\ldots,k-1$. □

3. Associated Stirling Numbers of the Second Kind

Now we consider the r-associated Stirling Numbers of the Second Kind, $S_r(n,k)$, $r\ge 1$, with a fix $k\ge 1$. The case with $k=1$ is trivial and we omit it.

Our first observation is that $\{k!S_r(n,k)\}_{n=1}^{\infty}$ does not seem to form a Newton–Euler Sequence for small values of k. Nevertheless, for the Newton–Euler Pair $(a_n,b_n=k!S_r(n,k))$ we have

Lemma 2. *For* $r\ge 2$ *and* $1\le j\le (r+2)k-1$, *we have*

$$a_j=\frac{k!S_r(j,k)}{j}$$

Proof. Similarly to the proof of Lemma 1, we also see that if $r\ge 2$ then $a_0=1$ and $a_j=k!S_r(j,k)/j$ for $j=1,2,\ldots,(r+2)k-1$. □

We note that it is easy to find the exponential generating function of the r-associated Stirling Numbers of the Second Kind $S_r(n,k)$

$$\sum_{n=0}^{\infty}S_r(n,k)\frac{x^n}{n!}=\frac{\left(e^x-\sum_{i=0}^{r-1}\frac{x^i}{i!}\right)^k}{k!}=\frac{\left(\sum_{i=r}^{\infty}\frac{x^i}{i!}\right)^k}{k!}$$

but this fact does not seem to help with the study of divisibility properties while the ordinary generating function appears to be more useful. First we mention some recurrences for these numbers and then move towards 2-adic divisibility properties. With $n\ge kr$ we have

$$S_r(n,k)=\binom{n-1}{r-1}S_r(n-r,k-1)+kS_r(n-1,k) \tag{3.6}$$

and

$$S_r(n,k) = \sum_{i=(k-1)r}^{n-r} \binom{n-1}{i} S_r(i,k-1). \tag{3.7}$$

We note that (3.6) and (3.7) are straightforward relations. We have the following lemma.

Lemma 3. *For $n \geq kr$, we have*

$$kS_r(n,k) = \sum_{i=(k-1)r}^{n-r} \binom{n}{i} S_r(i,k-1). \tag{3.8}$$

Proof. To prove (3.8), by subtracting (3.7) from (3.8), we consider the equivalent statement

$$(k-1)S_r(n,k) = \sum_{i=(k-1)r}^{n-r} \binom{n-1}{i-1} S_r(i,k-1)$$

and rewrite $S_r(i,k-1)$ by (3.6) as

$$\binom{i-1}{r-1} S_r(i-r,k-2) + (k-1)S_r(i-1,k-1).$$

With the identity $\binom{n-1}{i-1}\binom{i-1}{r-1} = \binom{n-1}{r-1}\binom{n-r}{i-r}$ and a few more steps, we can derive (3.8). $\square$

Now we find a lower bound on the 2-adic order of the Stirling numbers of the second kind. The case with $r = 1$ appeared in

Theorem 7. *(Theorem 3 in [2]) Let $n,k \in \mathbb{N}$ and $0 \leq k \leq n$. Then $\nu_2(S(n,k)) \geq d_2(k) - d_2(n)$.*

We can generalize this to associated Stirling numbers of the second kind.

Theorem 8. *Let $n,k \in \mathbb{N}$ and $0 \leq k \leq n$. Then*

$$\nu_2(S_r(n,k)) \geq d_2(k) - d_2(n)$$

and equivalently,

$$\nu_2(k!S_r(n,k)) \geq k - d_2(n).$$

Proof. Due to (3.8), the proof of Theorem 7 by induction on n can be easily adapted to Theorem 8 with $r \geq 1$. $\square$

Therefore, we can derive the following lower bounds from Theorem 8.

Corollary 1. *Let $c,k,n \in \mathbb{N}$ and $1 \leq k \leq 2^n$, then*

$$\nu_2(S_r(2^n,k)) \geq d_2(k) - 1 \tag{3.9}$$

and

$$\nu_2(S_r(c2^n,k)) \geq d_2(k) - d_2(c).$$

Remark 2. *Note that* (3.9) *is the best possible lower bound according to Conjecture 1 of Subsection 3.2.*

Our goal is to prove a statement similar to that of Theorem 4. We recall that the original approach in proving (1.2) relied on the inclusion–exclusion principle based relation

$$k!S(n,k)=\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n \tag{3.10}$$

in [1, (1b), p204] combined with Euler's theorem which helped in establishing the fact that $\nu_2(k!S(c2^n,k))$ is constant for any large enough n. However, if $r\geq 2$ then the inclusion–exclusion principle leads to a more complicated relation to which Euler's theorem cannot be directly applied. We will revisit this approach in Subsections 3.1 and 3.2 in some particular cases with $k=2,3,4$ for $r=2$ and $k=2,3$ for $r=3$, and in general.

3.1. The Case of $r=2$

For $r=2$, as a generalization of the generating function (2.4), the "vertical" generating function was partially given as

$$\sum_{n=0}^{\infty}S_2(n,k)x^n=\frac{x^{2k}\Phi_k(x)}{(1-x)^k(1-2x)^{k-1}\cdots(1-kx)} \tag{3.11}$$

in a slightly different form and explicitly determined for $1\leq k\leq 4$ in [10, identity (4.51)], where $\Phi_k(x)$ is a polynomial in x with integral coefficients of degree $k(k-1)/2$. For example,

$$\sum_{n=0}^{\infty}S_2(n,2)x^n=\sum_{n=4}^{\infty}S_2(n,2)x^n=\frac{x^4(3-2x)}{(1-x)^2(1-2x)}.$$

To be more precise, based on [10], we have that

$$\sum_{n=0}^{\infty}k!S_2(n,k)x^n=\sum_{n=2k}^{\infty}k!S_2(n,k)x^n=k!x^k\Psi_k(x) \tag{3.12}$$

with the recursively defined

$$\Psi_0(x)=1,\Psi_k(x)=\frac{kx\Psi_{k-1}(x)+x^2(\Psi_{k-1}(x))'}{1-kx},k=1,2,\ldots$$

which already guarantees the form (3.11) through partial fraction decomposition.

To test whether the sequence $\{k!S_r(n,k)\}_{n=1}^{\infty}$ forms a Newton–Euler Sequence we proceed as follows. By (3.12), we set

$$B_k(x)=k!x^k\Psi_k(x),$$

and

$$A_k(x)=e^{\int_c^x B_k(t)/t\,dt}$$

with some c and $A_k(0) = 1$. (The latter criterion can be satisfied if we set $c = 0$ or normalize by $a_0 = A_k(0)$.) Note that

$$\left(x^k\Psi_{k-1}(x)\right)' = x^{k-1}\left(k\Psi_{k-1}(x) + x(\Psi_{k-1}(x))'\right) = x^{k-2}(1-kx)\Psi_k(x),$$

thus,

$$\begin{aligned}\int B_k(x)/x\,dx &= k!\int x^{k-1}\frac{1}{x^{k-2}(1-kx)}\left(x^k\Psi_{k-1}(x)\right)'dx\\ &= k!\int \frac{x}{1-kx}\left(x^k\Psi_{k-1}(x)\right)'dx.\end{aligned}$$

We found that the above sequence is not a Newton–Euler Sequence for $k \le 9$ by finding a non-integer a_j of the Newton–Euler Pair $(a_n, b_n = k!S_2(n,k))$. Numerical evidence suggests that the smallest index j for which a_j is not an integer is increasing with k and it makes the testing computationally challenging.

Although we do not have congruences similar to (1.1) we can still address some divisibility properties. The sequences $\{k!S_2(n,k)\}_{n=1}^{\infty}$ with different values of k are included in [8], cf. columns of http://oeis.org/A200091. For instance, if $k = 2$, 3, and 4 then for the coefficients of $B_2(x) = \sum 2!S_2(n,2)x^n = 6x^4 + 20x^5 + 50x^6 + 112x^7 + 238x^8 + 492x^9 + 1002x^{10} + \ldots$, $B_3(x) = \sum 3!S_2(n,3)x^n = 90x^6 + 630x^7 + 2940x^8 + 11508x^9 + 40950x^{10} + \ldots$, and $B_4(x) = \sum 4!S_2(n,4)x^n = 2520x^8 + 30240x^9 + 226800x^{10} + \ldots$ we have that

$$2!S_2(n,2) = 2^n - 2n - 2, n \ge 4, \tag{3.13}$$

cf. http://oeis.org/A052515,

$$3!S_2(n,3) = 3^n - 3*2^{n-1}(n+2) + 3(n^2+n+1), n \ge 6, \tag{3.14}$$

cf. http://oeis.org/A224541 and http://oeis.org/A000478, and

$$4!S_2(n,4) = 4^n - 4*3^n - 4n3^{n-1} + 3n(n+3)*2^{n-1} + 6*2^n - 4n^3 - 8n - 4, n \ge 8, \tag{3.15}$$

http://oeis.org/A224542, respectively. Note that (3.13)-(3.15) can be easily derived by the inclusion–exclusion principle. It follows now that for $n \ge 4$

$$\nu_2(2!S_2(n,2)) = \nu_2(n+1) + 1$$

and $\nu_2(2!(S_2(c2^{n+1},2) - S_2(c2^n,2))) = n+1$ with c odd and $n \ge 2$, and for $n \ge 6$

$$\nu_2(3!S_2(n,3)) = \begin{cases} 2 & \text{if } n \equiv 0,1 \bmod 8,\\ 1 & \text{if } n \equiv 2,3,6,7 \bmod 8,\\ \nu_2(3^n + 3(n^2+n+1)) & \text{otherwise,}\end{cases}$$

and $\nu_2\left(3!(S_2(c2^{n+1},3) - S_2(c2^n,3))\right) = n$ with c odd and $n \ge 3$. For $n \ge 8$, we have

$$\nu_2(4!S_2(n,4)) = \begin{cases} 3 & \text{if } n \equiv 0,3 \bmod 8,\\ 4 & \text{if } n \equiv 1 \bmod 8,\\ 5 & \text{if } n \equiv 2 \bmod 8,\\ \nu_2(4*3^n + 4n3^{n-1} + 4n^3 + 8n + 4) & \text{otherwise,}\end{cases}$$

and $\nu_2\left(4!(S_2(c2^{n+1},4) - S_2(c2^n,4))\right) = n+2$ with c odd and $n \ge 3$.

Remark 3. *The basic difference among these cases is that, from the point of view of* 2*-adic valuation, the dominating terms in* (3.13)-(3.15) *have different forms. These terms are* $-2n$*,* $3n$*, and* $-4n3^{n-1}$ *for* $k = 2, 3$*, and* 4*, respectively.*

3.2. The Case of $r > 2$

We have only the special case with $r = 3$ and $k = 2$ that we could recognize in [8], cf. http://oeis.org/A052516. In fact, one can easily derive the generating function $\sum 2!S_3(n,2)x^n = 20x^6 + 70x^7 + 182x^8 + 420x^9 + 912x^{10} + \ldots$, with the coefficients $2!S_3(n,2) = 2^n - n^2 - n - 2$ for $n \geq 6$. It follows that for $n \geq 6$

$$\nu_2(2!S_3(n,2)) = \begin{cases} 1 & \text{if } 4 \mid n, \\ \nu_2(n^2+n+2) & \text{otherwise,} \end{cases}$$

and $\nu_2\left(2!(S_3(c2^{n+1},2) - S_3(c2^n,2))\right) = n$ with c odd and $n \geq 3$. Also note that the generating function is $2x^6(6x^2 - 15x + 10)/((1-x)^3(1-2x))$, cf. http://oeis.org/A052516.

For $r = k = 3$ we get that $3!S_3(n,3) = 3^n - 3 * 2^{n-3}(n^2 + 3n + 8) + 3/4(n^4 - 2n^3 + 7n^2 + 2n + 4)$ as well as

$$\nu_2(3!S_3(n,3)) = 2 \text{ if } 16 \mid n,$$

and $\nu_2\left(3!(S_3(c2^{n+1},3) - S_3(c2^n,3))\right) = n - 1$ with c odd and $n \geq 3$. The corresponding generating function is

$$\frac{6x^9(360x^6 - 2100x^5 + 5106x^4 - 6615x^3 + 4795x^2 - 1820x + 280)}{(1-x)^5(1-2x)^3(1-3x)}.$$

In case of a general r and k, we observe that

$$k!S_r(n,k) = \sum_{i=0}^{k} \sum_{t=0}^{k(r-1)} c_{t,i} \binom{n}{t} (k-i)^{n-t} \tag{3.16}$$

with integers $c_{t,i} = c_{t,i}(r,k)$ by the inclusion–exclusion principle. If $r \geq 2$ then this relation is significantly more complex than in the case of $r = 1$ as given in (3.10). We have not succeeded in using (3.16) to obtain $\nu_2(k!S_r(c2^n,k))$. Nonetheless, we use it to obtain some lower bound on $\nu_2\left(k!(S_r(c2^{n+1},k) - S_r(c2^n,k)\right)$. We apply a lemma from [5].

Lemma 4. *Let* $n, m \in \mathbb{N}$*, and* $c \geq 1$ *be an odd integer, then*

$$\nu_2\left((2m+1)^{c2^n} - 1\right) = n + 2 + \nu_2\left(\binom{m+1}{2}\right). \tag{3.17}$$

After further consideration, we observe that a general term in (3.16) has the form of

$$Cn^u(k-i)^{n-t} \tag{3.18}$$

with integer $u : 0 \leq u \leq t$ and a C such that $Ct!$ is an integer, i.e., $\nu_2(C) \geq -\nu_2(t!) = -t + d_2(t)$. By taking the largest value of t we get that $\nu_2(C) \geq -k(r-1) + d_2(k(r-1))$. (Note that it simplifies to $\nu_2(C) \geq 0$ if $r = 1$.) The form (3.18) holds true since $(n)_t = \binom{n}{t} t!$ is a polynomial in n with degree t and integral coefficients. Now we consider the general terms of the difference $k!(S_r(c2^{n+1},k) - S_r(c2^n,k))$. If $k - i = 2m+1$ is odd for some integer $m \geq 0$ then we get that $A = C(c2^{n+1})^u(2m+1)^{c2^{n+1}-t} - C(c2^n)^u(2m+1)^{c2^n-t} = C(c2^n)^u(2m+1)^{c2^n-t}\left(2^u(2m+1)^{c2^n} - 1\right)$. If $u = 0$ then $\nu_2(A) = \nu_2(C) + n + 2 + \nu_2\left(\binom{m+1}{2}\right)$ by Lemma 4. If $u \geq 1$ then $\nu_2(A) = \nu_2(C) + nu$.

The following lower bound follows.

Theorem 9. *Let $n,k,r \in \mathbb{N}$ and $c \geq 1$ be an odd integer, then*

$$\nu_2\left(k!(S_r(c2^{n+1},k) - S_r(c2^n,k))\right) \geq n - k(r-1) + d_2(k(r-1)).$$

We note that the exact 2-adic order can be determined if there is a unique 2-adically dominant term (with $u = 1$), cf. Remark 3 if $r = 2$ and the cases with $r = 3$ when it is $-n$ if $k = 2$ and $3n/2$ if $k = 3$, respectively. In Conjecture 2 we make a conjecture on the exact value.

Further numerical experimentation lead us to make the following conjectures in the style of Theorem 4 and Conjecture 2 of [5].

Conjecture 1. *Let $c,k,n,r \in \mathbb{N}$ and $1 \leq k \leq 2^n/r - 1$, then*

$$\nu_2\left(S_r(c2^n,k)\right) = d_2(k) - 1$$

or equivalently,

$$\nu_2\left(k!S_r(c2^n,k)\right) = k - 1.$$

Conjecture 2. *Let $n,k,r \in \mathbb{N}$, $2 \leq k \leq 2^n$, and $c \geq 1$ be an odd integer, then*

$$\nu_2\left(k!(S_r(c2^{n+1},k) - S_r(c2^n,k))\right) = n + g_r(k) \tag{3.19}$$

for some function $g_r(k)$ which is independent of n (for any sufficiently large n).

Remark 4. *Note that $g_2(2) = 1, g_2(3) = 0$, $g_2(4) = 2$, $g_3(2) = 0$ and $g_3(3) = -1$.*

3.3. General Formula for the Ordinary Generating Function of the r-associated Stirling Numbers of the Second Kind

We now derive a new general formula for the ordinary generating function of the sequence $\{k!S_r(n,k)\}_{n=1}^{\infty}$. We could not locate this generating function in the literature. In a similar fashion to formulas (3.11) and (3.12) we obtain

Theorem 10. *With some polynomial $\Lambda_{r,k}(x)$ in x with integral coefficients and of degree $(r-1)\binom{k}{2}$, we have*

$$\begin{aligned} B_{r,k}(x) &= \sum_{n=0}^{\infty} k!S_r(n,k)x^n \qquad (3.20)\\ &= \frac{k!x^{kr}\Lambda_{r,k}(x)}{(1-x)^{1+(k-1)(r-1)}(1-2x)^{1+(k-2)(r-1)}\cdots(1-kx)} \\ &= \frac{k!x^{kr}\Lambda_{r,k}(x)}{\prod_{i=1}^{k}(1-ix)^{1+(k-i)(r-1)}}. \end{aligned}$$

Proof. We set $C_{r,k}(x) = \sum_{n=0}^{\infty} S_r(n,k)x^n$ and note that $B_{r,k}(x) = k!C_{r,k}(x)$. We use the recurrence (3.6), multiply both sides by x^n and form a sum starting with the index $n = 0$. The sum on the left hand side is $C_{r,k}(x)$ while on the right hand side we get the sums

$$\begin{aligned}\sum_{n=0}^{\infty}\binom{n-1}{r-1}S_r(n-r,k-1)x^n &= \frac{x^r}{(r-1)!}\frac{d^{r-1}}{dx^{r-1}}\left(\sum_{n=0}^{\infty}S_r(n-r,k-1)x^{n-1}\right)\\ &= \frac{x^r}{(r-1)!}\frac{d^{r-1}}{dx^{r-1}}\left(x^{r-1}C_{r,k-1}(x)\right)\end{aligned}$$

and $kxC_{r,k}(x)$. This implies the recurrence relation

$$C_{r,k}(x) = \frac{1}{1-kx}\frac{x^r}{(r-1)!}\frac{d^{r-1}}{dx^{r-1}}\left(x^{r-1}C_{r,k-1}(x)\right). \tag{3.21}$$

We note that $C_{r,1}(x) = x^r/(1-x)$ and $\Lambda_{r,1}(x) = 1$ for $r \geq 1$, and now we can prove (3.20) by induction on k for each $r \geq 1$. The recurrence (3.21) and the nature of (3.20) suggests that we use the differentiation rule

$$\left(\prod_{i=0}^{k-1} f_i(x)\right)' = \left(\prod_{i=0}^{k-1} f_i(x)\right)\left(\sum_{i=0}^{k-1}\frac{f_i'(x)}{f_i(x)}\right) \tag{3.22}$$

with $f_0(x) = x^{r-1}\left(x^{(k-1)r}\Lambda_{r,k-1}(x)\right) = x^{kr-1}\Lambda_{r,k-1}(x)$ and $f_i(x) = 1/(1-ix)^{1+(k-1-i)(r-1)}, 1 \leq i \leq k-1$, in the partial fraction decomposition. Note that each term in $f_0'(x)$ has degree $kr-2+(r-1)\binom{k-1}{2}$ and in $f_0^{(r-1)}(x)$ has degree $(k-1)r+(r-1)\binom{k-1}{2}$ by the induction hypothesis. In fact, we can factor out $x^{(k-1)r}$ from each term, and after combining it with the factor x^r in (3.21), its contribution to $C_{r,k}(x)$ becomes the factor x^{kr}.

We observe that $f_i'(x)/f_i(x) = i(1+(k-1-i)(r-1))/(1-ix)$. By applying the relation (3.22) $r-1$ times, we gain the extra factor $((1-x)(1-2x)\ldots(1-(k-1)x))^{r-1}$ in the denominator of $C_{r,k}(x)$. For instance, we get that $C_{3,2}(x) = x^6(6x^2-15x+10)/((1-x)^3(1-2x))$ and $\Lambda_{3,2}(x) = 6x^2-15x+10$.

The degree of $\Lambda_{r,k}(x)$ follows by observing that $\Lambda_{r,1}(x) = 1$ and its degree increases by $(k-1)(r-1)$ due to the second factor of (3.22) as we move from $k-1$ to k. □

4. Conclusion

We proved that the Stirling Numbers of the Second Kind form Newton–Euler Sequences. This fact provides an alternative method to prove congruential properties. We also illustrated that the r-associated Stirling Numbers of the Second Kind do not form Newton–Euler Sequences. By another approach, some divisibility properties were presented or conjectured on the underlying quantities and their differences. We investigated some special cases with $r = 2$ and $k = 2,3,4$, as well as with $r = 3$ and $k = 2,3$. We also derived a new recurrence in k for the ordinary generating function $C_{r,k}(x) = \sum_{n=0}^{\infty} S_r(n,k)x^n = \sum_{n=kr}^{\infty} S_r(n,k)x^n$ of the associated Stirling Numbers of the Second Kind $\{S_r(n,k)\}_{n=1}^{\infty}$.

It appears that little is known about congruential and divisibility properties of the associated Stirling Numbers of the Second Kind. We derived two lower bounds on the 2-adic orders of these numbers and their differences. The new generating function (3.20) might help in finding exact p-adic properties for $p = 2$ and other primes too although the analysis of the generating function might need a new approach since the nature of its denominator seems significantly more complex than that of the ordinary Stirling Numbers of the Second Kind.

Our main results are summarized in Theorems 2, 8, 9, and 10. The Conjectures 1 and 2 suggest the exact 2-adic orders.

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