p-adic Properties of Recurrences Involving Stirling Numbers

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December 8, 2014

ABSTRACT: In 1984, Lengyel first introduced a recurrence involving Stirling numbers in his paper [Len84]. These numbers, known as the Lengyel numbers, have since been studied by many over the past 30 years. Interests with Lengyel numbers itself have evolved substantially, from the early interests in its asymptotic growth to the recent investigations of its p-adic properties. Indeed, other similarly defined combinatorial sequences have been defined, with the respective p-adic properties studied and recorded in various recent literature. In this article, we aim to give a brief survey on the p-adic investigations of these recurrences conducted over the years.

1 Introduction

In 1984, Lengyel posed in his paper [Len84] the following enumerative problem: given $n \in \mathbb{N}$, what is the number of not necessarily maximal chain in the partition lattice of $[n] = \{1, \cdots, n\}$ containing both the minimum and maximal element? These numbers, denoted by Z_n, are known as the Lengyel numbers. The enumerative and combinatorial properties of these numbers are well-understood since its introduction. For example, the Lengyel numbers form a sequence with a recursive structure which involves the Stirling numbers, and its corresponding exponential generating function form a nice combinatorial relation. We will describe the combinatorics of Lengyel numbers in Section 3.1.

The years following the introduction of Lengyel numbers saw a focus of research and investigations on these numbers towards its asymptotic growth rates, initiated by Lengyel [Len84], and later Lengyel used the Lengyel numbers as an example on some convergence results for recurrent sequences with Babai [BL92]. For more in depth discussion of these results, c.f. [Len84] and [BL92].

In the early 1990s, the p-adic studies of Stirling numbers were initiated by Clarke [Cla95] and Lengyel [Len94]. Particular emphasis was placed on the 2-adic properties of Stirling numbers of the second kind, perhaps due to the inherently applicability and concreteness of these numbers, which counts the number of ways to subdivide a set into a specific number of subsets. However, the machinery available at the time forces Lengyel to present his results with respect to some bound. These bounds is related to some results by Kwong [Kwo89] in 1989 on the minimal periodicity of the Stirling numbers of the second kind modulo 2^N, for $N \in \mathbb{N}$. At the end of his paper, Lengyel, motivated by results from numerical experimentation, conjectured that indeed, these bounds are not needed.

In 2005, De Wannemacker showed in [DW05] that such bounds are indeed redundant, and that he was able to prove the main result of Lengyel with no bounds involved. In 2009, Lengyel improved on the results of De Wannemacker, besides proving 2-adic properties of differences of Stirling Numbers of the second kind. We will outline the progression of these results in Section 2.2.

In general however, the p-adic properties of Stirling numbers of the second kind are quite difficult to derive. One may need to build some sophisticated machinery and employ in the studies of the p-adic property of Stirling numbers of the second kind, much like what is done by Berrizbeitia, Medina, Moll, Moll, and Noble in [BMM+10]. However, one can still derive some partial results of the p-adic properties of Stirling number of the second kind. We will discuss these in Section 2.3.

Whilst the Lengyel numbers are defined recursively with Stirling numbers of the second kind, only the 2-adic properties of Stirling number of the second kind seemed to yield some partial results on the 2-adic properties of Lengyel numbers, most of them described in Lengyel's [Len12]. Instead, Barsky and Bézivin showed in [BB14] how one can convert Lengyel numbers to be defined recursively with Stirling numbers of the first kind instead. With some result they have on the p-adic valuation of Stirling numbers of the first kind, they were able to reprove Lengyel's results and verify some conjecture of Lengyel. The results on the 2-adic properties of Lengyel numbers are described in Section 3.2.

However, Lengyel proved some results on the divergence rate of the p-adic valuation of Stirling numbers of the first kind in [Len15], and this makes generalisation into results on the p-adic properties of Lengyel numbers difficult. Indeed, Lengyel [Len12] remarked that it may be difficult, simply because there are not apparent structure on the p-adic valuation of Lengyel numbers.

Even so, Barsky and Bézivin described in [BB14] a generalisation of Lengyel numbers that does yield some result on its p-adic property under similar analysis as in the 2-adic case. In particular, they introduced a sequence of p-adic integers, which we refer to as the generalised Lengyel numbers in this paper, which yields as the main result surveyed by this paper much the same p-adic properties as the 2-adic properties of Lengyel numbers. We will describe the generalised Lengyel numbers and its p-adic properties in Section 4.

Finally, we will conclude this discussion by pointing out some interesting open problems to show that there are still work to be done to further our understanding of the p-adic properties of Lengyel numbers and its generalisation.

2 Stirling Numbers

In the theory of enumerative combinatorics, Stirling numbers are a celebrated class of combinatorial numbers which are fundamental and influential in many different areas of enumerative combinatorics. Named after the Scottish mathematician James Stirling in the 18-th century, there are indeed two different kinds of Stirling numbers, known as the Stirling numbers of the first kind and Stirling numbers of the second kind. Here, we will provide a brief overview of some combinatorial properties of both Stirling numbers. We refer the interested reader to the excellent book on combinatorics, "Enumerative Combinatorics, Vol. 1" by Richard Stanley [Sta11] for a thorough discussion on the combinatorics of Stirling numbers. We will then discuss the p-adic properites of Stirling numbers of both the first and second kind.

2.1 Combinatorics of Stirling Numbers

Consider the set $[n] := \{1, 2, \cdots, n\}$. One can define Stirling numbers of the second kind as follows:

Definition 2.1. *The number of ways to partition $[n]$ into k non-empty subsets, denoted by $S(n, k)$, is defined as the **Stirling numbers of the second kind**.*

An alternate way to define the Stirling numbers of the second kind is that they are the set of numbers such that

$$x^n = \sum_{k=0}^{\infty} S(n,k)(x)_k$$

where we define $(x)_k = k!\binom{x}{k} = x(x-1)\cdots(x-k+1)$ for $k \in \mathbb{N}\backslash\{0\}$ and $(x)_0 = 1$. It is well known that Stirling numbers of the second kind satisfy the following recurrence

$$S(n,k) = kS(n-1,k) + S(n-1,k-1)$$

Another well-known identity of the Stirling numbers of the second kind can be deduced by using the inclusion-exclusion principle

$$k!S(n,k) = \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}i^n$$

A perhaps less well-known identity on the Stirling numbers of the second kind is the following result by De Wannemacker [DW05], which occurs in De Wannemacker's studies of calculations in the Witt ring.

Lemma 2.1. *Let $n, k, m \in \mathbb{N}$ be such that $0 \leqslant k \leqslant n+m$. Then*

$$S(n+m,k) = \sum_{i=0}^{k}\sum_{j=i}^{k}\binom{j}{i}\frac{(k-i)!}{(k-j)!}S(n,k-i)S(m,j)$$

Proof. Recall the Chu-Vandermonde identity

$$\binom{s+t}{q} = \sum_{r=0}^{q}\binom{s}{r}\binom{t}{q-r}$$

where $q, r, s, t \in \mathbb{N}$. Then, for $n, m \in \mathbb{N}$, we get that

$$\begin{aligned}
x^{n+m} = x^n x^m = \sum_{r=0}^{n} S(n,r)(x)_r \left(\sum_{j=0}^{m} S(m,j)j!\binom{x}{j}\right) &= \sum_{r=0}^{n} S(n,r)(x)_r \left(\sum_{j=0}^{m} S(m,j)j!\sum_{i=0}^{j}\binom{x-r}{i}\binom{r}{j-i}\right) \\
&= \sum_{r=0}^{n} S(n,r)\sum_{j=0}^{m} S(m,j)\sum_{i=0}^{j}\frac{j!}{i!}\binom{r}{j-i}(x)_{r+i} \\
&= \sum_{r=0}^{n}\sum_{i=0}^{m}\sum_{j=i}^{m}\frac{j!}{i!}\binom{r}{j-i}S(n,r)S(m,j)(x)_{r+i} \\
&\qquad \text{(rearranging the sums)} \\
&= \sum_{r=0}^{n}\sum_{i=0}^{m}\sum_{j=i}^{r+i}\frac{j!}{i!}\binom{r}{j-i}S(n,r)S(m,j)(x)_{r+i} \\
&\qquad \text{(since } S(m,j)=0 \text{ for } j \geqslant m+1\text{)} \\
&= \sum_{k=0}^{n+m}\sum_{i=0}^{k}\sum_{j=i}^{k}\frac{j!}{i!}\binom{k-i}{j-i}S(n,k-i)S(m,j)(x)_k \\
&\qquad \text{(substituting } k=r+i\text{)}
\end{aligned}$$

Thus, by comparing coefficients, we get that for $0 \leqslant k \leqslant n+m$,

$$S(n+m,k) = \sum_{i=0}^{k}\sum_{j=i}^{k}\frac{j!}{i!}\binom{k-i}{j-i}S(n,k-i)S(m,j) \qquad \square$$

Now, let S_n be the set of all permutations on $[n]$. We can then define the Stirling numbers of the first kind:

Definition 2.2. *The* ***unsigned Stirling numbers of the first kind****, $c(n,k)$, is defined to be*

$$c(n,k) = |\{\sigma \in S_n;\ \sigma \text{ has } n \text{ cycles}\}|$$

and the ***(signed) Stirling numbers of the first kind*** *is $s(n,k) = (-1)^{n-k}c(n,k)$.*

Similar (or perhaps inverse) to Stirling numbers of the second kind, Stirling numbers of the first kind have the following alternate definition:

$$(x)_n = \sum_{k=1}^{n} s(n,k)x^k$$

Stirling numbers of the first kind has identities similar to those for Stirling number of the second kind. For example, the unsigned Stirling number of the first kind is known to have the following recurrence structure

$$c(n+1,k) = nc(n,k) + c(n,k-1)$$

and so this immediately implies that

$$s(n+1,k) = -ns(n,k) + s(n,k-1)$$

As the name suggests, Stirling numbers of the first and second kind are closely related. In particular, they satisfy for all $n, m \in \mathbb{N}$,

$$\sum_{k=0}^{n} S(n,k)s(k,m) = \delta_{n,m}; \ \sum_{k=0}^{n} s(n,k)S(k,m) = \delta_{n,m}$$

where $\delta_{n,m}$ is the Kronecker delta, and further they satisfy the Stirling inversion formula: $\forall n \in \mathbb{N}$,

$$a_n = \sum_{k=0}^{n} s(n,k)b_k \iff b_n = \sum_{k=0}^{n} S(n,k)a_k$$

where $\{a_m\}_{m\geqslant 0}$ and $\{b_m\}_{m\geqslant 0}$ are sequences of elements of a field K. This particular relation between both Stirling numbers is what will allow us to derive the general result in the study of the p-adic properties of recurrences involving these numbers.

2.2 2-adic Properties of Stirling Numbers of the Second Kind

Let us recall the following well-known lemma of Legendre [Leg30]: For any $n \in \mathbb{N}$, p a prime, we have that

$$\mathrm{ord}_p(n!) = \frac{n - d_p(n)}{p-1}$$

where $d_p(n)$ is the sum of the digits of n in its base p representation. In particular, this means that $\mathrm{ord}_2(n!) = n - d_2(n)$. From this, it is also immediate to deduce that for all $n, k \in \mathbb{N}$ and p a prime

$$\mathrm{ord}_p\left(\binom{n}{k}\right) = \mathrm{ord}_p(n!) - \mathrm{ord}_p(k!) - \mathrm{ord}_p((n-k)!) = \frac{d_p(k) + d_p(n-k) - d_p(n)}{p-1}$$

In 1994, Lengyel in [Len94] studied the 2-adic properties of the Stirling numbers of the second kind. Building on Nijenhuis' and Wilf's [NW87] and Kwong's [Kwo89] works on the studies of the minimum periodicities of Stirling numbers of the second kind, Lengyel was able to prove some 2-adic properties of Stirling numbers of the second kind. In particular, let $k \in \mathbb{N}$, and let p be a prime. For $N \geqslant 1$, let $\pi(k, p^N)$ and $n_0(k, p^N)$ be the minimum period and the smallest number of non-repeating terms respectively of the sequence of Stirling numbers $\{S(n,k) \bmod p^N\}_{n\geqslant k}$. We define the following function

$$f(c,x) = \max\left\{d_2(k) + \lceil \log_2(k) \rceil + k, \left\lceil \frac{\log_2(n_0(k, 2^{d_2(k)}))}{c} \right\rceil\right\}$$

then, we get the following result of Lengyel:

Theorem 2.1. *For all positive integers c, k and n such that $n \geqslant f(c,k)$,*

$$\mathrm{ord}_2(S(c \cdot 2^n, k)) = d_2(k) - 1$$

Lengyel also proceeded to prove the 2-adic properties of Stirling numbers of the second kind of the form $S(c \cdot 2^n + u, k)$, for various value of u including the case $u = -1$. These results are essentially derived from the above theorem, and as such require a similar bound on n. C.f. [Len94] for a detailed derivation of these results. At the end of his paper [Len94], Lengyel conjectured that indeed, at least in the case where $u = 0$, these bounds are redundant with respect to the study of the 2-adic order of Stirling numbers of the second kind.

In 2005, De Wannemacker finally proved in his paper [DW05] that these bounds are indeed redundant. In particular, he showed the following theorem:

Theorem 2.2. *Let $n, k \in \mathbb{N}$ be such that $1 \leqslant k \leqslant 2^n$. Then*

$$\mathrm{ord}_2(S(2^n, k)) = d_2(k) - 1$$

For brevity, we will provide only a sketch of the proof. For the full proof, c.f. [DW05]. For the proof, we need the following lemma of De Wannemacker.

Lemma 2.2. *Let $m, n \in \mathbb{N}$. Then,*

$$d_2(m+n) \leqslant d_2(m) + d_2(n)$$

Further, let $\varepsilon_l(k)$ be the coefficient of 2^l in the binary expansion of k, for all $l \geqslant 0$. Then

$$d_2(m+n) = d_2(m) + d_2(n) \iff \sum_{l \geqslant 0} \varepsilon_l(m)\varepsilon_l(n) = 0$$

Proof. (Sketch proof of Theorem 2.2) We will proceed by induction on n. If $n = 0$, then

$$\mathrm{ord}_2(S(1,1)) = \mathrm{ord}_2(1) = 0 = d_2(1) - 1$$

Suppose the theorem is true for all $i \leqslant n-1$, for some $n \in \mathbb{N}$. If $k = 1$, then again,

$$\mathrm{ord}_2(S(2^n, 1)) = \mathrm{ord}_2(1) = 0 = d_2(1) - 1$$

so assume $k > 1$. By Lemma 2.1, we get that

$$S(2^n, k) = \sum_{i=0}^{k} \sum_{j=i}^{k} \binom{j}{i} \frac{(k-i)!}{(k-j)!} S(2^{n-1}, k-i) S(2^{n-1}, j)$$

Now, for each summand, by induction hypothesis and Legendre identities and its corollary,

$$\begin{aligned}
&\mathrm{ord}_2\left(\binom{j}{i}\frac{(k-i)!}{(k-j)!} S(2^{n-1}, k-i) S(2^{n-1}, j)\right) \\
&= \mathrm{ord}_2\left(\binom{j}{i}\right) + \mathrm{ord}_2((k-i)!) - \mathrm{ord}_2((k-j)!) + \mathrm{ord}_2(S(2^{n-1}, k-i)) + \mathrm{ord}_2(S(2^{n-1}, j)) \\
&= d_2(i) + d_2(j-i) - d_2(j) + (k-i) - d_2(k-i) - (k-j) + d_2(k-j) + d_2(k-i) - 1 + d_2(j) - 1 \\
&= d_2(i) + d_2(j-i) + j - i + d_2(k-j) - 2
\end{aligned}$$

By Lemma 2.2, since $d_2(m) \leqslant d_2(m-n) + d_2(n) \implies d_2(m-n) \geqslant d_2(m) - d_2(n)$ for $n \geqslant m$,

$$d_2(i) + d_2(j) - d_2(i) + j - i + d_2(k) - j) - 2 \geqslant d_2(i) + d_2(j) - d_2(i) + j - i + d_2(k) - d_2(j) - 2 = d_2(k) + j - i - 2$$

and since $j \geqslant i$, this shows that the 2-adic valuation of each summand is at least $d_2(k) - 2$. If the 2-adic valuation of a summand is greater than or equal to $d_2(k)$, then these summands are divisible by $2^{d_2(k)-1}$. For summands which have 2-adic valuation $d_2(k) - 2$ or $d_2(k) - 1$, we can proceed by doing a case analysis on these summands. The following table gives the coefficients of these summands

	Coefficient of $2^{d_2(k)-2}$	**Coefficient of $2^{d_2(k)-1}$**	**Final Coefficient of $2^{d_2(k)-1}$**
$d(k) = 1$	0	odd	odd
$d_2(k) > 1$ & k odd	$2\times$ odd	even	odd
$d_2(k) > 1$ & k even	$2\times$ odd	even	odd

where the last column is the coefficient of $2^{d_2(k)-1}$ by grouping all the terms above. Thus

$$S(2^n, k) = C \cdot 2^{d_2(k)-1} + \sum_{j \geqslant d_2(k)} C_j \cdot 2^j$$

where C is odd, and C_j for $j \geqslant d_2(k)$ are some non-negative integers. Thus, we get that $\mathrm{ord}_2(S(2^n, 1)) = d_2(k) - 1$, and the result follows by induction. $\square$

De Wannemacker also proved in [DW05] a lower bound for all other cases.

Theorem 2.3. *Let $n, k \in \mathbb{N}$ be such that $0 \leqslant k \leqslant n$. Then,*

$$\operatorname{ord}_2(S(n,k)) \geqslant d_2(k) - d_2(n)$$

Subsequently, in 2009, Lengyel generalised in his paper [Len09] the results of De Wannemacker.

Theorem 2.4. *Let $n, k, c \in \mathbb{N}$, and let $1 \leqslant k \leqslant 2^n$. Then,*

$$\operatorname{ord}_2(S(c \cdot 2^n, k)) = d_2(k) - 1$$

In essence, the above theorem could be proved by performing induction on $d_2(c)$, and applying the techniques employed by De Wannemacker. Indeed, among other results regarding the 2-adic and p-adic properties of differences of Stirling numbers of the second kind, Lengyel proved the following bound.

Theorem 2.5. *Let $n, k, u, c \in \mathbb{N}$ with $1 \leqslant k \leqslant 2^n$ and $u \leqslant 2^{\operatorname{ord}_2(k)}$. Then*

$$\operatorname{ord}_2(S(c \cdot 2^n + u, k)) \geqslant \operatorname{ord}_2(k) - \lfloor \log_2(u) \rfloor + d_2(k) - 2$$

and furthermore, if $u = 2^m$, for some $m \in \mathbb{N}$ with $0 \leqslant m \leqslant \operatorname{ord}_2(k) - 1$, then

$$\operatorname{ord}_2(S(c \cdot 2^n + 2^m, k)) = \operatorname{ord}_2(k) - m + d_2(k) - 2$$

2.3 p-adic Properties of Stirling Numbers of the Second Kind

In general, for $p > 2$, it is generally much harder to consolidate the p-adic properties of Stirling numbers of the second kind. One will need to employ what is known as modular and p-adic trees to derive some general results on the p-adic properties of Stirling numbers of the second kind. We refer the reader to a paper by Berrizbeitia et. al. [BMM+10] for more results on the p-adic valuation of Stirling numbers of the second kind using this machinery. We do, however, have the following result by Gessel and Lengyel [GL01] regarding the p-adic properties of Stirling numbers of the second kind, without using the machinery employed by Berrizbeitia et. al. in their study of the p-adic valuation of Stirling numbers of the second kind.

Theorem 2.6. *If $n = a(p-1)p^q$, where $1 \leqslant k \leqslant n$, with a and q co-prime positive integers, q sufficiently large, and $\frac{k}{p}$ not an odd integer, then*

$$\operatorname{ord}_p(S(n,k)) = \left\lfloor \frac{k-1}{p-1} \right\rfloor + \tau_p(k) - \frac{k - d_p(k)}{p-1}$$

where $\tau_p(k)$ is a nonnegative integer. Furthermore, if $p - 1 \mid k$, then $\tau_k(p) = 0$.

2.4 p-adic Properties of Stirling Numbers of the First Kind

Curiously, although there are many results regarding the p-adic properties of Stirling numbers of the second kind, the same could not be said for the p-adic properties of Stirling numbers of the first kind until very recently. This is especially curious, since some of our general result we will describe later in this paper on the p-adic properties of recurrences involving Stirling numbers rely heavily on the p-adic properties of Stirling numbers of the first kind, even when they are defined with Stirling numbers of the second kind.

It wasn't until very recently that Lengyel showed why such is the case, when he proved in [Len15] some divergence results of the p-adic valuation of Stirling numbers of the first kind. Nonetheless, the following lemma due to Barsky and Bézivin in [BB14] will still prove useful to us.

Lemma 2.3. *For $1 \leqslant k \leqslant n$, we get that*

$$\operatorname{ord}_p(s(n,k)) \geqslant \left\lfloor \frac{n-1}{p} \right\rfloor + 1 - k$$

Proof. Recall that $(x)_n = \sum_{k=1}^{n} s(n,k)x^k$. Let $P_n(x) = (x)_n \in \mathbb{C}_p[x]$. From [Ami75],

$$|P_n|_p(x) = \sup_{\substack{x \in \mathbb{C}_p \\ |x|_p \leqslant r}} |P_n(x)|_p$$

Consider $\alpha \in \mathbb{C}_p$ with $|\alpha|_p = r$, with $\frac{1}{p} < r < 1$. Then, since $|\cdot|_p$ is non-Archimedian,

$$|\alpha - a|_p = \begin{cases} 1 & p \nmid a \\ r & \text{otherwise} \end{cases}$$

Since the number of elements of $1 \leqslant a \leqslant n-1$ which are divisible by p is $\left\lfloor \frac{n-1}{p} \right\rfloor$, this implies that

$$|(\alpha)_n|_p = \left|\alpha(\alpha-1)\cdots(\alpha-(n-1))\right|_p = |\alpha|_p|\alpha-1|_p \cdots |\alpha-(n-1)|_p = r \cdot r^{\left\lfloor \frac{n-1}{p} \right\rfloor} = r^{\left\lfloor \frac{n-1}{p} \right\rfloor + 1}$$

But note also that

$$r^{\left\lfloor \frac{n-1}{p} \right\rfloor + 1} = |(\alpha)_n|_p = \max_{1 \leqslant k \leqslant n} \{|s(n,k)|_p r^k\}$$

Thus, we get that $|s(n,k)|_p r^k \leqslant r^{\left\lfloor \frac{n-1}{p} \right\rfloor + 1} \implies |s(n,k)|_p \leqslant r^{\left\lfloor \frac{n-1}{p} \right\rfloor + 1 - k}$. Further, for $x \in \mathbb{C}_p$ such that $|x|_p < r$,

$$\begin{cases} |x-a|_p = 1 & p \nmid a \\ |x-a|_p \leqslant \max\{|x|_p, |a|_p\} < r & \text{otherwise} \end{cases}$$

$$\implies |P_n(x)|_p = \left|x(x-1)\cdots(x-(n-1))\right|_p = |x|_p|x-1|_p \cdots |x-(n-1)|_p < r \cdot r^{\left\lfloor \frac{n-1}{p} \right\rfloor} = r^{\left\lfloor \frac{n-1}{p} \right\rfloor + 1}$$

Thus, $|P_n|_p(x) = r^{\left\lfloor \frac{n-1}{p} \right\rfloor + 1}$. Finally, since the map

$$r \mapsto |P_n|_p(r)$$

is a continuous map [Ami75], this allows us to let $r = \frac{1}{p}$, i.e. we get that

$$|s(n,k)|_p \leqslant \frac{1}{p^{\left\lfloor \frac{n-1}{p} \right\rfloor + 1 - k}} \implies \mathrm{ord}_p(s(n,k)) \geqslant \left\lfloor \frac{n-1}{p} \right\rfloor + 1 - k \qquad \square$$

Finally, we note that [Len15] saw some new results on the p-adic relation for the generalised harmonic numbers, besides results on the divergence rate of the p-adic valuation of Stirling numbers of the first kind.

3 Lengyel Numbers

In 1984, Lengyel first introduced a recurrence involving Stirling numbers in his paper [Len84]. These numbers, known as the Lengyel numbers, have since been studied by many over the past 30 years. Interests with Lengyel numbers itself have evolved substantially, from the early interests in its asymptotic growth to the recent investigations of its p-adic properties. Here, we will first give the combinatorial properties of Lengyel numbers. We will then describe the 2-adic properties of Lengyel numbers, and discuss the difficulty of deriving p-adic properties of Lengyel numbers for $p > 3$. We will also introduce a Lengyel-like sequence of numbers, which instead of Stirling number of the second kind, has a recurrence structure that is defined by Stirling number of the first kind.

3.1 Combinatorics of Lengyel Numbers

Consider the set $[n] = \{1, 2, \cdots, n\}$. Let P_n be the set of all possible partitions of $[n]$ into k non-empty subsets, for $k \in \mathbb{N}$.

Definition 3.1. *Let $x, y \in P_n$. We say x is a* ***refinement*** *of y, denoted by $x \preceq y$, if every element of x is a subset of an element in y.*

$\preceq$ can easily be checked to be a partial order. We can then define a partition lattice, by imposing the partial order $\preceq$ on P_n. Specifically,

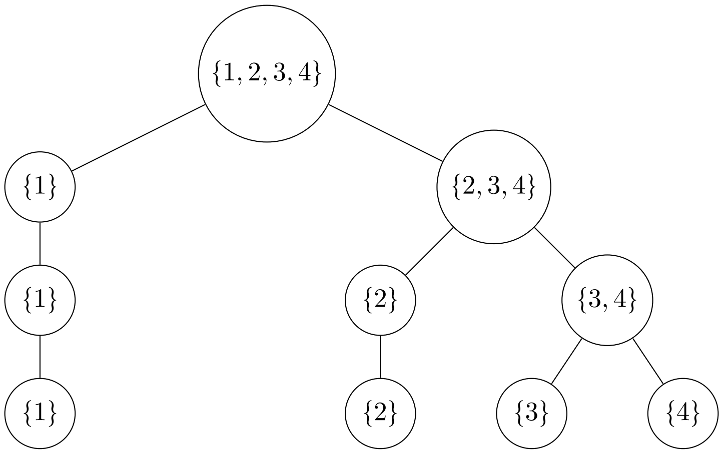
Definition 3.2. *Let $n \geqslant 1$, and let P_n be the set of all possible partitions of $[n]$ into k non-empty subsets, for $k \in \mathbb{N}$. The* ***partition lattice*** *of $[n]$ is then $\mathcal{L}_n = (P_n, \preceq)$, where $\preceq$ is the refinement partial order on P_n.*

It is now easy to see that the minimal and maximal element of $\mathcal{L}_n$ is $\{\{1\},\{2\},\cdots,\{n\}\}$ and $\{[n]\}$ respectively. Now, since $\mathcal{L}_n$ is a poset, we can consider the chains of $\mathcal{L}_n$ containing both the minimal and maximal element. If we request the chain to be maximal, by basic counting the number of maximal chains in $\mathcal{L}_n$ is $\frac{n!(n-1)!}{2^{n-1}}$.

Instead, let us consider the number of not necessarily maximal chains in $\mathcal{L}_n$ containing both the minimal and maximal element, which we denote by Z_n. We can represent any such chain by the tree representation, where the nodes at the n-th level from the bottom of the tree represents the element of the n-th partition in the chain, and any two nodes x and y are adjacent if either $x \subseteq y$ or $y \subseteq x$. For example, if $n = 4$, then one possible chain is the chain

$$\{\{1\},\{2\},\{3\},\{4\}\} \preceq \{\{1\},\{2\},\{3,4\}\} \preceq \{\{1\},\{2,3,4\}\} \preceq \{1,2,3,4\}$$

then, it's tree representation is



Now, it is clear that $Z_1 = 1$. Indeed, we have the following recurrence structure.

Lemma 3.1. *Let $n \geqslant 2$, and let Z_n be the number of not necessarily maximal chain in $\mathcal{L}_n$ containing both the minimal and maximal element of $\mathcal{L}_n$. Then,*

$$Z_n = \sum_{k=1}^{n-1} S(n,k) Z_k$$

To see why this true, view each chain in its tree representation. Note that we can decompose the tree representation into the bottom level and subtree $\tilde{T} = T \backslash \{\{1\},\cdots,\{n\}\}$, i.e. $\tilde{T}$ is the tree obtained by deleting the bottom level nodes. Now, by viewing each node as the atomic element, $\tilde{T}$ is a tree representation of a chain in $\mathcal{L}_k$, where k is the number of bottom level nodes of $\tilde{T}$. Thus, there are Z_k such tree representations, and in each there are $S(n,k)$ ways to partition $[n]$ into k non-empty subsets for each of the k nodes of $\tilde{T}$. We can do this for each $1 \leqslant k \leqslant n$, and so we get the above total.

From this recurrence, one can easily deduce that for all $n \geqslant 2$,

$$2Z_n = \sum_{k=1}^{n} S(n,k) Z_k$$

Now, if we define the following exponential generating function,

$$Z(x) = \sum_{n \geqslant 1} Z_n \frac{x^n}{n!}$$

then, the above recurrence yields the functional equation

$$2Z(x) - x = Z(e^x - 1)$$

Indeed, if one is familiar with the theory of combinatorial species, one can impose the species $\mathcal{Z}$, which endows given any $[n]$, a tree structure on $[n]$ which is the tree representation of not necessarily maximal partition-chain containing both the minimal and maximal element of $\mathcal{L}_n$. Then, $Z(x)$ defined above is the exponential generating function for $\mathcal{Z}$. Further, if $\mathcal{E}$, $\mathcal{E}_0$ and $\mathcal{E}_1$ is the species of sets, the empty set and the singleton respectively, then we get the following natural equivalence

$$\mathcal{Z} \oplus \mathcal{Z} \simeq \mathcal{Z}[\mathcal{E} \backslash \mathcal{E}_0] \oplus \mathcal{E}_1$$

and the theory of combinatorial species will immediately allow us to deduce the above functional equation. The functional equation has been used to derive results about the asymptotic growth rates of the Lengyel numbers, but has yet to be used to deduce the p-adic properties of these numbers.

3.2 2-adic Properties of Lengyel Numbers

Lengyel, perhaps appropriately, is the first to describe some 2-adic properties of the Lengyel numbers, in his paper [Len12]. The main result of [Len12] describes a very beautiful lower bound for the 2-adic valuation of the Lengyel numbers.

Theorem 3.1. *For $n \geqslant 2$ and $L \geqslant 0$, we have that*

$$\mathrm{ord}_2(Z_{2^n + L}) \geqslant n$$

Note that this theorem immediately implies that for any $k \geqslant 4$,

$$\mathrm{ord}_2(Z_k) \geqslant \lfloor \log_2(k) \rfloor$$

and since $\mathrm{ord}_2(Z_1) = 0$, $\mathrm{ord}_2(Z_2) = 0$ and $\mathrm{ord}_2(Z_3) = 2$, we get that for all $k \geqslant 1$,

$$\mathrm{ord}_2(Z_k) \geqslant \lceil \log_2(k) \rceil - 1 \geqslant \log_2(k) - 1$$

In [Len12], Lengyel used a variety of results regarding the 2-adic valuation of different sequences of Stirling number of the second kind and its difference he derived earlier in [Len09] as well as in [Len12] to prove Theorem 3.1. At the end of [Len12], Lengyel proposed a couple of conjectures regarding the exact 2-adic valuation of some Lengyel numbers, and suggests some other conjectures regarding the 2-adic valuation of Stirling number of the second kind in some form to prove. Lengyel was, under the assumption of these conjectures regarding the 2-adic valuation of Stirling number of the second kind, able to provide a conditional prove of his conjectures on the exact 2-adic valuation for $k = 2^n$ for $n \geqslant 3$, and the maximum k such that $\mathrm{ord}_2(Z_k) = n$. See [Len12] for more details.

In 2014, Barsky and Bézivin, in their paper [BB14], considered the Lengyel numbers in terms of Stirling numbers of the first kind. Indeed, since for $n \geqslant 2$,

$$2Z_n = \sum_{k=1}^{n} S(n,k) Z_k = \sum_{k=1}^{n} S(n,k) Z_k + \delta_{n,1} = \sum_{k=1}^{n} S(n,k) Z_k + \sum_{k=1}^{n} S(n,k) s(k,1) = \sum_{k=1}^{n} S(n,k)(Z_k + s(n,1))$$

$$\implies Z_n = \sum_{k=1}^{n} S(n,k) \frac{Z_k + s(n,1)}{2}$$

by applying the Stirling inversion formula, we get that

Proposition 3.2. *For all $n \geqslant 2$, we get that*

$$Z_n = s(n,1) - 2\sum_{k=1}^{n-1} s(n,k)Z_k$$

With this insight, Barsky and Bézivin was able to reprove (the slight relaxation of) Theorem 3.1, i.e.

$$\operatorname{ord}_2(Z_k) \geqslant \log_2(k) - 1$$

We will prove a generalisation of this in the next section. Further, they were able to resolve the conjectures of Lengyel in [Len12] without proving Lengyel's conjectures regarding the 2-adic valuation of Stirling number of the second kind. Specifically,

Theorem 3.3. *If $t \geqslant 3$, then*

$$\operatorname{ord}_2(Z_{2^t}) = t$$

Theorem 3.4. *For all $t \geqslant 2$,*

$$\max\{k;\ \operatorname{ord}_2(Z_k) = t\} = 3 \cdot 2^{t-1}$$

For a full proof, c.f. [BB14].

3.3 p-adic Properties of Lengyel Numbers

For $p \neq 2$, Lengyel noted in [Len12] that through his numerical experimentation, that the p-adic valuation of Lengyel numbers, $\operatorname{ord}_p(Z_k)$ does not seem to have any structure, and indeed that it behaves quite chaotically. Lengyel suggested that perhaps the functional equation $2Z(x) + x = Z(e^x - 1)$ may help in deriving the p-adic properties of Lengyel numbers.

Perhaps, one should instead try to generalise the Lengyel numbers in some way to get results that are similar as in the case of the 2-adic properties of Lengyel numbers. Indeed, we will introduce in the next section, a generalisation of the Lengyel numbers due to Barsky and Bézivin that yields a similar p-adic properties for any $p \neq 2$ as the 2-adic properties of Lengyel numbers.

3.4 Lengyel-like Numbers With Stirling Numbers of the First Kind

Barsky and Bézivin considered in [BB14] a new sequence Y_n by swapping out Stirling numbers of the second kind with Stirling numbers of the first kind. Specifically, the sequence Y_n is such that $Y_1 = 1$ and for $n \geqslant 2$,

$$Y_n = \sum_{k=1}^{n-1} s(n,k)Y_k$$

We can then apply the same argument as in for Proposition 3.2 to see that

$$Y_n = -S(n,1) + 2\sum_{k=1}^{n} S(n,k)Y_k$$

Barsky and Bźivin proved the following 2-adic properties of Y_n.

Theorem 3.5. *For all $n \geqslant 1$, Y_n is odd. Further,*

$$\operatorname{ord}_2(Y_n + 1) = 1 \iff n \equiv 0 \text{ or } 1 \bmod 3$$

For a proof, see [BB14].

4 p-adic Properties of A Generalisation of Lengyel Numbers

In [BB14], Barsky and Bézivin introduced a generalisation of Lengyel numbers, which will be referred as the generalised Lengyel numbers in this paper, although this terminology was not used in [BB14].

Definition 4.1. *Let p be a prime. A* ***generalised Lengyel number*** *is a sequence of numbers $Z_n^{\langle p\rangle}$ defined as follows: Define $Z_0^{\langle p\rangle} = 0$, $Z_1^{\langle p\rangle} = 1$. For $n \geqslant 2$, define*

$$(p-1)Z_n^{\langle p\rangle} = \sum_{k=1}^{n-1} S(n,k) Z_k^{\langle p\rangle}$$

From this, it is clear that although $Z_n^{\langle p\rangle}$ is not an integer, it is a p-adic integer. Furthermore, with the same reasoning as in Proposition 3.2, we get that for $n \geqslant 2$,

$$Z_n^{\langle p\rangle} = s(n,1) - \frac{p}{p-1}\sum_{k=1}^{n-1} s(n,k) Z_k^{\langle p\rangle}$$

This is indeed a generalisation of Lengyel numbers, as if $p=2$, we get that $Z_1^{\langle 2\rangle} = 1 = Z_1$, and so inductively,

$$Z_n^{\langle 2\rangle} = \sum_{k=1}^{n-1} S(n,k) Z_k^{\langle 2\rangle} = \sum_{k=1}^{n-1} S(n,k) Z_k = Z_n$$

Before we proceed to describe the main result, let us first observe the following lemma due to Barsky and Bézivin [BB14].

Lemma 4.1. *Let $t \geqslant 2$. Then*

$$(x)_{p^t} \equiv x^{p^{t-1}}(x^{p-1}-1)^{p^{t-1}} \bmod p$$

Remark 4.1: Now, since

$$\sum_{k=1}^{p^t} s(n,k)x^k = (x)_{p^t} \equiv x^{p^{t-1}}(x^{p-1}-1)^{p^{t-1}} \equiv x^{p^{t-1}}(x^{p^t-p^{t-1}}-1) \equiv x^{p^t} - x^{p^{t-1}} \bmod p$$

This implies by comparing coefficients that $p \mid s(n,k)$ for $k \neq p^{t-1}, p^t$ and that $p \nmid s(n, p^{t-1})$.

With this generalisation and our results from previous section, we can now prove the main theorem due to Barksky and Bézivin in [BB14].

Theorem 4.2. *For all $n \geqslant 1$,*

$$\mathrm{ord}_p(Z_n^{\langle p\rangle}) \geqslant \log_p(n) - 1$$

and further, if $p \geqslant 3$, then for $t \geqslant 1$

$$\mathrm{ord}_p(Z_{p^t}^{\langle p\rangle}) = t-1$$

Proof. Since

$$Z_n^{\langle p\rangle} = s(n,1) - \frac{p}{p-1}\sum_{k=1}^{n-1} s(n,k) Z_k^{\langle p\rangle}$$

$$\implies \mathrm{ord}_p(Z_n^{\langle p\rangle}) \geqslant \min\left\{\mathrm{ord}_p(s(n,1)), \mathrm{ord}_p\left(\frac{p}{p-1}s(n,1)Z_1^{\langle p\rangle}\right), \cdots, \mathrm{ord}_p\left(\frac{p}{p-1}s(n,n-1)Z_{n-1}^{\langle p\rangle}\right)\right\}$$

Thus, we need only show that p-adic valuation of each summand is at least $\log_p(n) - 1$. We will proceed by induction on n. For $1 \leqslant n \leqslant p$, $\log_p(n) - 1 \leqslant 0$, so the result holds trivially. Now, suppose the result holds for all $m \leqslant n-1$, for some $n \in \mathbb{N}$ and $n \geqslant p+1$.

Now, recall that $s(n,1) = (-1)^{n-1}(n-1)!$, and by Legendre's lemma, $\text{ord}_p(s(n,1)) = \frac{n-1-d_p(n-1)}{p-1}$. Write $n-1$ in its base-p expansion,

$$n - 1 = \sum_{j=0}^{m} a_j p^j$$

for some $m \geqslant 0$, with $a_j \in \{0, \cdots, p-1\}$ for $0 \leqslant j \leqslant m$ and $a_m \neq 0$. Thus implies that

$$n = 1 + \sum_{j=0}^{m} a_j p^j \leqslant 1 + \sum_{j=0}^{m} (p-1)p^j = 1 + (p-1)\sum_{j=0}^{m} p^j = 1 + p^{m+1} - 1 = p^{m+1}$$

$$\implies \log_p(n) - 1 \leqslant m$$

Finally,

$$\begin{aligned} \text{ord}_p(s(n,1)) = \frac{n-1-d_p(n-1)}{p-1} &= \frac{\sum_{j=0}^{m} a_j p^j - \sum_{j=0}^{m} a_j}{p-1} = \frac{\sum_{j=1}^{m} a_j (p^j - 1)}{p-1} \\ &= \sum_{j=1}^{m} a_j \sum_{i=0}^{j-1} p^i \geqslant a_m(1 + \cdots + p^{m-1}) \geqslant m \geqslant \log_p(n) - 1 \end{aligned}$$

For each summand $\frac{p}{p-1} s(n,k) Z_k^{\langle p \rangle}$,

$$\text{ord}_p\left(\frac{p}{p-1} s(n,k) Z_k^{\langle p \rangle}\right) = \text{ord}_p\left(ps(n,k) Z_k^{\langle p \rangle}\right) = 1 + \text{ord}_p(s(n,k)) + \text{ord}_p(Z_k^{\langle p \rangle})$$

By Lemma 2.3 and by induction hypothesis, we get that

$$\text{ord}_p\left(\frac{p}{p-1} s(n,k) Z_k^{\langle p \rangle}\right) = 1 + \text{ord}_p(s(n,k)) + \text{ord}_p(Z_k^{\langle p \rangle}) \geqslant 1 + \left\lfloor \frac{n-1}{p} \right\rfloor + 1 - k + \log_p(k) - 1 = 1 + \left\lfloor \frac{n-1}{p} \right\rfloor - k + \log_p(k)$$

Now, for $1 \leqslant k \leqslant \left\lfloor \frac{n-1}{p} \right\rfloor + 1$, there are two cases:

Case 1: $p \mid n$. Then, $n = mp \implies \log_p(n) = \log_p(m) + 1$, for some $m \in \mathbb{N}$, and that $\left\lfloor \frac{n-1}{p} \right\rfloor = m - 1$. So, $1 \leqslant k \leqslant m$. Now, recall that $f(x) = x - \log_p(x)$ is an increasing function, so we get that $k - \log_p(k) \leqslant m - \log_p(m)$, and thus

$$\text{ord}_p\left(\frac{p}{p-1} s(n,k) Z_k^{\langle p \rangle}\right) \geqslant 1 + (m-1) - k + \log_p(k) \geqslant m - m + \log_p(m) = \log_p(n) - 1$$

Case 2: $p \nmid n$. Then, $n = mp + r \implies \log_p(n) \geqslant \log_p(m) + 1$, for some $m \in \mathbb{N}$, and that $\left\lfloor \frac{n-1}{p} \right\rfloor = m$ and $1 \leqslant k \leqslant m + 1$. Now, recall that $f(x) = x - \log_p(x)$ is an increasing function, so we get that $k - \log_p(k) \leqslant m + 1 - \log_p(m+1)$, and thus

$$\text{ord}_p\left(\frac{p}{p-1} s(n,k) Z_k^{\langle p \rangle}\right) \geqslant 1 + m - k + \log_p(k) \geqslant 1 + m - m - 1 + \log_p(m) = \log_p(n) - 1$$

Finally, let $\left\lfloor \frac{n-1}{p} \right\rfloor + 2 \leqslant k \leqslant n - 1$. Now, $\left\lfloor \frac{n-1}{p} \right\rfloor < k \implies pk > n - 1$, or that $pk \geqslant n$. Thus $\log_p(k) \geqslant \log_p(n) - 1$, and so

$$\text{ord}_p\left(\frac{p}{p-1} s(n,k) Z_k^{\langle p \rangle}\right) \geqslant 1 + \text{ord}_p(Z_k^{\langle p \rangle}) \geqslant \log_p(k) \geqslant \log_p(n) - 1$$

Thus, we get that

$$\mathrm{ord}_p(Z_n^{\langle p\rangle}) \geqslant \min\left\{\mathrm{ord}_p(s(n,1)), \mathrm{ord}_p\left(\frac{p}{p-1}s(n,1)Z_1^{\langle p\rangle}\right), \cdots, \mathrm{ord}_p\left(\frac{p}{p-1}s(n,n-1)Z_{n-1}^{\langle p\rangle}\right)\right\} \geqslant \log_p(n) - 1$$

and we get our result by induction.

For the second part of the theorem, let $p \geqslant 3$, and suppose $n = p^t$. We proceed by induction on t. For $t = 1$, since

$$Z_p^{\langle p\rangle} = s(p,1) - \frac{p}{p-1}\sum_{k=1}^{p-1} s(n,k)Z_k^{\langle p\rangle}$$

and $s(p,1) = (-1)^{p-1}(p-1)!$, so $Z_p^{\langle p\rangle}$ is a p-adic unit, which implies that $\mathrm{ord}_p(Z_p^{\langle p\rangle}) = 0$.

Now, suppose the result is true for $k \leqslant t-1$, for some $2 \leqslant t \in \mathbb{N}$. Again, since $s(p^t,1) = (-1)^{p^t-1}(p^t-1)!$,

$$\mathrm{ord}_p(s(p^t,1)) = \frac{p^t - 1 - d_p(p^t-1)}{p-1} = \frac{p^t - 1 - t(p-1)}{p-1} = 1 + \cdots + p^{t-1} - t = (p-1) + \cdots + (p^{t-1}-1)$$

where each summand is at least 1, and further since $p \geqslant 3$, $p - 1 \geqslant 2$, and so

$$\mathrm{ord}_p(s(p^t,1)) = (p-1) + \cdots + (p^{t-1}-1) \geqslant t$$

Now, suppose $1 \leqslant k \leqslant p^{t-1} - 2$. Since $p \geqslant 3$ and $t \geqslant 2$, $p^{t-1} - 2 \geqslant p^{t-2} \implies \log_p(p^{t-1}-2) \geqslant t-2$. Thus, since $f(x) = x - \log_p(x)$ is increasing,

$$k - \log_p(k) \leqslant (p^{t-1}-2) - \log_p(p^{t-1}-2) \leqslant (p^{t-1}-2) - (t-2) = p^{t-1} - t$$

and since $1 + \left\lfloor \frac{p^t-1}{p} \right\rfloor = 1 + p^{t-1} - 1 = p^{t-1}$,

$$\implies \mathrm{ord}_p\left(\frac{p}{p-1}s(n,k)Z_k^{\langle p\rangle}\right) \geqslant 1 + \left\lfloor \frac{p^t-1}{p} \right\rfloor - k + \log_p(k) = p^{t-1} - k + \log_p(k) \geqslant t$$

For $k = p^{t-1} - 1$, since by remark 4.1, we have that $p \mid s(p^t, p^{t-1}-1) \implies \mathrm{ord}_p(s(p^t, p^{t-1}-1) \geqslant 1$, so by first part of the theorem,

$$p^{t-1} - 1 > p^{t-2} \implies \mathrm{ord}_p(s(p^t,p^{t-1}-1)) > t-3 \implies \mathrm{ord}_p(s(p^t,p^{t-1}-1)) \geqslant t-2$$

$$\implies \mathrm{ord}_p\left(\frac{p}{p-1}s(n,p^{t-1}-1)Z_{p^{t-1}-1}^{\langle p\rangle}\right) = 1 + \mathrm{ord}_p(s(p^t,p^{t-1}-1)) + \mathrm{ord}_p(Z_{p^{t-1}-1}^{\langle p\rangle}) \geqslant 2 + t - 2 = t$$

If $p^{t-1} + 1 \leqslant k \leqslant p^t - 1$, then by first part of the theorem, $\mathrm{ord}_p(Z_k^{\langle p\rangle}) \geqslant t-1$, so

$$\mathrm{ord}_p\left(\frac{p}{p-1}s(n,k)Z_k^{\langle p\rangle}\right) \geqslant 1 + \mathrm{ord}_p(Z_k^{\langle p\rangle}) \geqslant t$$

Finally, for $k = p^{t-1}$, by remark 4.1, we have that $p \nmid s(p^t, p^{t-1}-1)$, so by induction hypothesis, since $\mathrm{ord}_p(Z_{p^{t-1}}^{\langle p\rangle}) = t-2$,

$$\mathrm{ord}_p\left(\frac{p}{p-1}s(n,p^{t-1})Z_{p^{t-1}}^{\langle p\rangle}\right) = 1 + \mathrm{ord}_p(s(p^t,p^{t-1})) + \mathrm{ord}_p(Z_{p^{t-1}}^{\langle p\rangle}) = t-1$$

Thus, since every other summand has p-adic valuation greater than $t-1$, we must have that

$$\mathrm{ord}_p(Z_{p^t}^{\langle p\rangle}) = t-1$$

and so the result follows by induction as required. $\square$

5 Conclusion

In the past 30 years since the introduction of Lengyel numbers by Lengyel in 1984, the interest in Lengyel numbers has changed drastically from its asymptotic growth rate to its p-adic properties. Along the way to finally showing the main theorem of Barsky and Bézivin, a lot of results, particularly the p-adic properties of Stirling numbers, has been discovered that are beautiful in their own right. However, these results really do come together and give a really elegant lower bound to the p-adic valuation of generalised Lengyel numbers. This particular problem really appeals to the root of p-adic analysis, which is the question of the divisibility by primes of numbers. Further, this problem hints perhaps at a fascinating link between combinatorics and p-adic analysis, suggesting that maybe other classical combinatorial objects yields similar results when subjected to a similar analysis.

Of course, this problem is far from being entirely solved, as there are still some open questions such as the possible p-adic properties of Lengyel numbers for $p \neq 2$ and other refinement on the bounds of the p-adic valuation of generalised Lengyel numbers. It is also interesting, as Lengyel has pointed out, to see if the theory of species may help yield more result on the p-adic properties of these sequence. Finally, it remains to see if the generalised Lengyel numbers has some concrete interpretation in combinatorics or even in other areas of mathematics, and if $\mathrm{ord}_q(Z_n^{\langle p \rangle})$ yields any structure, for $p \neq q$.

Acknowledgements

The author of this paper wishes to express gratitude to Professor Cameron L. Stewart, of the University of Waterloo, for the wonderful opportunity to work in this interesting problem. The author will also like to thank Hao (Billy) Lee, of the University of Waterloo, for his suggestions and recommendations while the author presented a summary of this paper in a talk.

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