ARTICLE IN PRESS

Applied Mathematics Letters ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

The conditional gambler's ruin problem with ties allowed

Tamás Lengyel

Occidental College, 1600 Campus Road, Los Angeles, CA 90041, USA

ARTICLE INFO

Article history:
Received 20 April 2007
Received in revised form 14 March 2008
Accepted 25 April 2008

Keywords:
Gambler's ruin
Generating function
Lattice path combinatorics
Continued fraction representation by polynomials
Chebyshev polynomials of the second kind

ABSTRACT

We determine the distribution of duration in the gambler's ruin problem given that one specific player wins. In this version we allow ties in the single games. We present a unified approach which uses generating functions to prove and extend some results that were obtained in [Frederick Stern, Conditional expectation of the duration in the classical ruin problem, Math. Mag. 48 (4) (1975) 200–203; S.M. Samuels, The classical ruin problem with equal initial fortunes, Math. Mag. 48 (5) (1975) 286–288; W.A. Beyer, M.S. Waterman, Symmetries for conditioned ruin problems, Math. Mag. 50 (1) (1977) 42–45].

© 2008 Elsevier Ltd. All rights reserved.

1. Introduction

In the classical gambler's ruin problem we are interested in the probability of ruin and the probability distribution of the duration of the game (and sometimes, of first-passage times); cf. [4]. The results are well known. But what if we want to answer these questions under the assumption that one specific player wins?

In the gambler's ruin problem, two players play a series of single games in which one dollar changes hands until one player goes bankrupt. Let s (start) and g (goal) be positive integers with $s < g$. One player (to whom we refer as "our player") starts with an initial capital of s dollars while the other player has $g - s$ dollars, and in each single game our player either wins \$1 with probability p or loses it with probability q. We will also allow ties with probability $r = 1 - p - q$ when no money is changed. We will assume that $0 < p, q < 1$ and $0 \le r < 1$.

In the language of Markov chains, there are two absorbing states, 0 and g. Our player starts at state s, $1 \le s \le g - 1$, and the game ends when this player first reaches either state 0 or g. Let S_n be the capital of our player after n games and N be the duration of the game. Thus, $S_0 = s$ and S_N is either 0 or g. We note that the case with $s = m$ and $g = 2m$ corresponds to the popular winning-a-series-by-m-games scheme in sports (often combined with a requirement on the minimum number of games to be won) [5].

In Section 2, we exhibit generating function based proofs of some facts regarding the conditional gambler's ruin problem with ties allowed in the single games. Section 3 outlines approaches for obtaining the expected duration of the conditional game for $p = q \le 1/2$. Results concerning some games with several players are summarized in Section 4 indicating that these games exhibit characteristics different from those of the two-player games.

2. Some generalizations: Conditioning and different goals

Conditioning random walks with absorbing boundaries attracted some attention in the seventies in the context of cancer growth modeling [6]. In particular, there was significant interest in estimating the expected time for a cancerous clone of

E-mail address: lengyel@oxy.edu.

0893-9659/\$ – see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2008.04.007

Please cite this article in press as: T. Lengyel, The conditional gambler's ruin problem with ties allowed, Appl. Math. Lett. (2008), doi:10.1016/j.aml.2008.04.007

cells to reach tumor size from a single wayward cell, given that the tumor size is reached, rather than dying off before reaching tumor size. The analysis can help to determine the appearance of the first wayward cancerous cell, and perhaps, to identify reasons for the cancerous growth.

Let A denote the event that our player wins, i.e., $S_N = g$, with $S_0 = s$ and $S_i \neq 0$ and g, for all $i : 0 < i < N$. The distribution of the conditional duration is $P(N = n \mid A)$ with $n = g - s, g - s + 1, \ldots$.

Stern [1] observed a surprising symmetry:

Theorem 1 (*Stern*). *The expected conditional durations are equal for p, $0 < p < 1$, and $q = 1 - p$, given that the walk starts at the middle, i.e., $s = m$ and $g = 2m$.*

Samuels [2] improved this theorem and got the stronger result in

Theorem 2 (*Samuels*). *Under the conditions of Theorem 1, the duration N and the end point S_N, i.e., who wins, are independent.*

Beyer and Waterman [3] generalized Stern's result to the case when the two players may have different goals, i.e., if they start with not necessarily equal initial fortunes, e.g., $g - s \neq s$, by using combinatorial arguments. They found that the transition probabilities of the walk conditioned on our player winning depend on the actual position and are symmetric in p and $q = 1 - p$ [3, Theorem]. They also proved that this implies

Theorem 3 (*Beyer and Waterman*). *The duration conditioned on our player winning, $P(N = n \mid A)$, is symmetric in p and q.*

After stating the generalizations of these theorems to the case in which ties are allowed, we will prove them by using generating functions.

Theorem 4. *The expected conditional durations are equal for p, $0 < p < 1$, and $q = 1 - p - r$, given that the walk starts at the middle, i.e., $s = m$ and $g = 2m$.*

Theorem 5. *Under the conditions of Theorem 4, the duration N and the end point S_N, i.e., who wins, are independent.*

Theorem 6. *The duration conditioned on our player winning, $P(N = n \mid A)$, is symmetric in p and q, even if ties are allowed.*

Clearly, the theorems are true if $p = q$; thus, we assume that $p \neq q$ from now on. We need some preliminaries.

We set $U_{s,g}(x, p) = \sum_{n=0}^{\infty} P(N = n, A)\, x^n$, i.e., the generating function of the probability of winning the series by our player at the nth single game, and $f_{s,g}(x, p) = \sum_{n=0}^{\infty} P(N = n)\, x^n$ the unconditional probability generating function of the duration. (For the sake of notational simplicity, we indicate only the probability parameter p of winning in a single game rather than all three parameters p, q, and r but will explicitly include the other parameters where necessary, e.g., $U_{s,g}(x, p)$ stands for $U_{s,g}(x, p, q, r)$.) It is easy to see that the probability of winning by this player is

$$P(A) = U_{s,g}(1, p) = \frac{1 - \left(\frac{q}{p}\right)^s}{1 - \left(\frac{q}{p}\right)^g},$$

and it remains true even if ties are allowed. Indeed, if we want to determine the winning probabilities only, we can ignore games that end in a tie, and therefore, we can use

$$p' = \frac{p}{p+q} \quad \text{and} \quad q' = \frac{q}{p+q} \tag{1}$$

to denote the probability of winning and losing a game, respectively, given that the game is not a tie. This implies that

$$\frac{1 - \left(\frac{q'}{p'}\right)^s}{1 - \left(\frac{q'}{p'}\right)^g} = \frac{1 - \left(\frac{q}{p}\right)^s}{1 - \left(\frac{q}{p}\right)^g}.$$

Now we follow an approach outlined in [4, Chapter XIV] to find the generating function of the probability of winning at the nth single game with ties being allowed. This generating function can be written as

$$U_{s,g}(x, p) = A(x, p)\lambda_1^s(x, p) + B(x, p)\lambda_2^s(x, p), \quad 0 \le s \le g,$$

with the boundary conditions $U_{0,g}(x, p) = 0$ and $U_{g,g}(x, p) = 1$. Note that $\lambda_1(x, p) = \frac{1 - rx + \sqrt{(1-rx)^2 - 4pqx^2}}{2px}$ and $\lambda_2(x, p) = \frac{1 - rx - \sqrt{(1-rx)^2 - 4pqx^2}}{2px}$ are the solutions to the underlying characteristic equation which is based on the recurrence relation

$$U_{s,g}(x, p) = pxU_{s+1,g}(x, p) + rxU_{s,g}(x, p) + qxU_{s-1,g}(x, p), \quad 1 \le s \le g - 1.$$

After determining that $A(x,p) = -B(x,p) = \left(\lambda_1^g(x,p) - \lambda_2^g(x,p)\right)^{-1}$, we get

$$U_{s,g}(x,p) = \frac{\lambda_1^s(x,p) - \lambda_2^s(x,p)}{\lambda_1^g(x,p) - \lambda_2^g(x,p)}, \quad 0 \le s \le g. \tag{2}$$

We note that for the unconditional case, the conditions on f are identical to those on U except that the boundary condition $f_{0,g}(x,p) = 1$ replaces $U_{0,g}(x,p) = 0$. Thus the probability generating function of the duration is

$$f_{s,g}(x,p) = \frac{\lambda_1^s(x,p)(1-\lambda_2^g(x,p)) - \lambda_2^s(x,p)(1-\lambda_1^g(x,p))}{\lambda_1^g(x,p) - \lambda_2^g(x,p)}, \tag{3}$$

for all $s : 0 \le s \le g$.

We are now ready to prove the generalization of Samuels' result, Theorem 5, using generating functions. Theorem 4 then follows immediately.

Proof of Theorem 5. In fact, we want to check whether

$$\sum_{n=0}^{\infty} P(N=n, A)\, x^n = P(A) \sum_{n=0}^{\infty} P(N=n)\, x^n,$$

i.e., whether N and S_N are independent random variables. We now prove that this is true if $s = m$ and $g = 2m$. By identities (2) and (3), we get that

$$\begin{aligned}\frac{\sum_{n=0}^{\infty} P(N=n)x^n}{\sum_{n=0}^{\infty} P(N=n,A)x^n} &= \frac{f_{s,g}(x,p)}{U_{s,g}(x,p)}\\ &= 1 + (\lambda_1(x,p)\lambda_2(x,p))^s \frac{\lambda_1^{g-s}(x,p) - \lambda_2^{g-s}(x,p)}{\lambda_1^s(x,p) - \lambda_2^s(x,p)}.\end{aligned} \tag{4}$$

Since $\lambda_1(x,p)\lambda_2(x,p) = q/p$, this implies that the ratio in (4) is

$$1 + \left(\frac{q}{p}\right)^s = \frac{1}{P(A)}.$$

Therefore, in this case the probability generating function f is exactly $1/P(A)$ times the generating function $U_{s,g}$. Note that the ratio is free of x.

On the other hand, it is easy to see that if $s \ne g - s$ then the ratio in (4) is not identically equal to a constant for all p, $0 < p < 1$. ■

Proof of Theorem 6. We prove that $P(N = n \mid A)$ is symmetric in p and q. By the definition of $\lambda_1(x,p)$ and $\lambda_2(x,p)$, we get that

$$\lambda_1^k(x,p) - \lambda_2^k(x,p) = \frac{(2qx)^k}{(2px)^k}\left(\lambda_1^k(x,q) - \lambda_2^k(x,q)\right)$$

for any $k \ge 0$. According to (2), this implies that

$$\frac{U_{s,g}(x,p,q,r)}{U_{s,g}(x,q,p,r)} = \frac{p^{g-s}}{q^{g-s}} = \frac{1-\left(\frac{q}{p}\right)^s}{1-\left(\frac{q}{p}\right)^g}\,\frac{1-\left(\frac{p}{q}\right)^g}{1-\left(\frac{p}{q}\right)^s} = \frac{U_{s,g}(1,p,q,r)}{U_{s,g}(1,q,p,r)},$$

which is free of x. By comparing the coefficients of the terms of x^n in $U_{s,g}(x,p,q,r)/U_{s,g}(1,p,q,r)$ and $U_{s,g}(x,q,p,r)/U_{s,g}(1,q,p,r)$, it follows that

$$P(N = n \mid A) = \frac{P(N=n, A)}{P(A)}$$

is the same when p and q are exchanged. ■

Please cite this article in press as: T. Lengyel, The conditional gambler's ruin problem with ties allowed, Appl. Math. Lett. (2008), doi:10.1016/j.aml.2008.04.007

3. The expected conditional duration

We can also generalize Stern's other result [1] on the expected duration conditioned on our player winning in the case of $p = q$. The purpose of this section is to show how to obtain this result by other methods, and in particular, by determining the generating function of the duration by means of lattice path combinatorics. The latter approach might help to extend the investigation to games with $p \neq q$.

Theorem 7 *(Stern). With $p = q = 1/2$ and $E_{s,g}(1/2) = E(N \mid A)$, we have that*

$$\frac{s}{g}E_{s,g}(1/2) = \lim_{x\to 1^-} U'_{s,g}(x, 1/2) = \frac{s}{g}\frac{g^2 - s^2}{3} \tag{5}$$

where $U'_{s,g}(x,p)$ stands for $\frac{\partial}{\partial x}U_{s,g}(x,p)$.

One of the simplest nontrivial examples for the conditioned gambler's ruin has $s = 1$ and $g = 3$ with $r = 0$. The letters W and L stand for winning and losing \$1, respectively. (For short, we write $E_{s,g}$ in place of $E_{s,g}(1/2)$.)

Example 1. With the usual symbolic setting and translation (e.g., [7] or [8]) we get for the generating function that

$$(WpLq)^*WpWp \Rightarrow U_{1,3}(x,p) = \frac{p^2x^2}{1 - pqx^2}$$

which becomes $U_{1,3}(x, 1/2) = \frac{x^2/4}{1-x^2/4}$, and thus, $\frac{1}{3}E_{1,3} = 8/9$ if $p = q = 1/2$.

We note the relation between the unconditional and conditional expected durations $E(N) = \frac{s}{g}E_{s,g} + \frac{g-s}{g}E_{g-s,g}$. It immediately implies that $E(N)$ is 2 in the above example.

Proof of Theorem 7. This can be proven by using harmonic functions or generating functions. The former is based on the fact that $a_{s,g} = \frac{s}{g}E_{s,g} + \frac{s^3}{3g}$ with $a_{0,g} = 0$ and $a_{g,g} = \frac{g^2}{3}$ is a harmonic function in $s : 0 < s < g$. We can easily see that $a_{s,g} = \frac{sg}{3}$ which implies (5).

The latter approach works by approximating $U_{s,g}(1-\varepsilon, 1/2)$ and $U'_{s,g}(1-\varepsilon, 1/2)$ about $\varepsilon = 0$ or by symbolic tools. For example, the first-order approximation yields $U_{s,g}(1-\varepsilon, 1/2) = \frac{s}{g} - \frac{s(g^2-s^2)}{3g}\varepsilon + O(\varepsilon^2)$, while the *Mathematica* command `Series[D[`$\frac{\lambda_1[x,1/2]^s - \lambda_2[x,1/2]^s}{\lambda_1[x,1/2]^g - \lambda_2[x,1/2]^g}$`, {x, 1}]/.x` $\to 1-\varepsilon$, `{`ε`, 0, 0}] //Simplify` implies (5). ∎

The above proof can be easily generalized to games with ties allowed:

Theorem 8. *With $p = q = \frac{1-r}{2}$ and $E_{s,g}\left(\frac{1-r}{2}\right) = E(N \mid A)$, we have that*

$$\frac{s}{g}E_{s,g}\left(\frac{1-r}{2}\right) = \lim_{x\to 1^-} U'_{s,g}\left(x, \frac{1-r}{2}\right) = \frac{s}{g}\frac{g^2 - s^2}{3}\frac{1}{1-r} \tag{6}$$

where $U'_{s,g}(x,p)$ stands for $\frac{\partial}{\partial x}U_{s,g}(x,p)$.

Another proof of Theorem 8. The conditional duration can be written as a random sum of N' independent and identically distributed random variables; the ith is made of an arbitrary nonnegative number T_i, $i = 1, 2, \ldots, N'$, of ties immediately before the ith win or loss followed by the win or loss, i.e.,

$$N = \sum_{i=1}^{N'}(T_i + 1). \tag{7}$$

The distribution of $T_i + 1$ is geometric with parameter $1 - r$, and thus, by Wald's identity and Theorem 7, we get that $\frac{s}{g}E_{s,g}(1/2) \cdot \frac{1}{1-r}$, i.e., identity (6). ∎

An alternative derivation of the generating function is based on the correspondence between games and paths by the theory of lattice path combinatorics; cf. [7]. Flajolet and Guillemin developed an approach in terms of a fundamental continued fraction and its associated convergent polynomials to construct the underlying generating function. We illustrate this approach by sketching the proof of (5) for the symmetric game with $p = q = 1/2$ and $s = 1$. Readers are advised to consult [7] for technical details.

Using identities for the generating function of upcrossings with standard "combinatorial morphism" [7, Sections 2.3–4], we have for $g \geq 2$ that

$$U_{1,g}(x, 1/2) = \frac{A_{g-1}(x)}{Q_{g-1}(x)},$$

with $A_g(x) = (x/2)^g$ and, thus, $A'_g(1) = g/2^g$, $g \geq 0$. At the core of this method are the polynomials $Q_h = Q_h(x)$ that satisfy the second-order recurrence

ARTICLE IN PRESS

$$Q_{h+1} = Q_h - a_{h-1}b_hQ_{h-1}, \quad h \geq 0, \tag{8}$$

with $a_{i-1} = b_i = x/2$, $i \geq 1$, the convention $a_{-1}b_0 = 1$, and the initial conditions $Q_{-1} = 0$ and $Q_0 = 1$ [7, Definition 1]. Determining these polynomials via (8) is instrumental in the analysis of the generating function.

The main observation is that we are able to determine $Q_g(1)$ and $Q_g'(1)$ without completely constructing $Q_g(x)$. In fact, we set the generating function

$$q(t,x) = \sum_{n=0}^{\infty} Q_n(x)t^n$$

of the polynomials $Q_n(x)$, $n \geq 0$, and get that

$$q(t,x) = \frac{1}{1 - t + \frac{x^2}{4}t^2}. \tag{9}$$

Note that here $q(2t/x, x)$ is the generating function of the Chebyshev polynomials (with argument x) of the second kind at $1/x$; cf. [9].

To make things a little simpler, we observe that $q(2t, 1) = (1-t)^{-2}$ and $\frac{\partial}{\partial x}q(2t, 1) = -\frac{2t^2}{(1-t)^4}$ which yield that $Q_g(1) = \frac{g+1}{2^g}$ and $Q_g'(1) = -\frac{g(g^2-1)}{3 \cdot 2^g}$. Putting everything together we get that $U'_{1,g}(1, 1/2) = \frac{g^2-1}{3g}$.

4. Several players

We note that the gambler's ruin problem can be generalized to $K \geq 2$ players. We include some interesting facts showing succinct differences from the two-player setting even if ties are not allowed and the winning probabilities are equal.

In the generalized setting two or more players have equal initial resources but possibly different winning probabilities, p_i, $i = 1, 2, \ldots, K$. In one of the popular versions, in every single game, one player, say the ith player, is randomly chosen to be the winner with probability p_i, and one dollar is paid to the winner by each of the other $K-1$ players. Typically, ties are not considered. The duration until the first or all but one player go bankrupt is analyzed. Note that for $K \geq 3$ players, it is possible for more than one player to be ruined at the same time, at the first ruin time. Rocha and Stern [10] found that the independence of the duration N until first ruin and which player is ruined can be extended to K-player asymmetric games, $K \geq 2$, with equal initial fortunes m provided that $1 \leq m \leq K+1$. Here an asymmetric game means that the individual single game winning probabilities can be different for the K players. Somewhat surprisingly, the independence breaks down for $K = 3$ and $m = 5$, even for the symmetric game, as was shown in [11]. We note that for games with equal initial fortunes m, the duration of the game must be congruent to $m \bmod K$, and $P(N = m) = 1$ for $1 \leq m < K$, and $P(\frac{N-m \bmod K}{K} = t)$ decreases exponentially as the nonnegative integer $t \to \infty$, for $m \geq K$ [11, Theorem 3]. They also considered asymptotic results for the expected duration and, in the case of symmetric games, for the individual and combined ruin probabilities provided that $m - K$ remains fixed as $K \to \infty$ [11, Theorems 4 and 5].

In general, after renormalization similar to (1), the winning probabilities are not affected by allowing ties as ties can be ignored from the point of view of winning. Most results regarding the expected duration $E(N)$ can be extended by the method of the proof of Theorem 8 via the summation (7).

Acknowledgments

The author wishes to thank David Collins for raising some of the questions, Gregory Tollisen and the referee, in particular, for making helpful suggestions on the organization of the work and comments that improved its presentation.

References

[1] Frederick Stern, Conditional expectation of the duration in the classical ruin problem, Math. Mag. 48 (4) (1975) 200–203.
[2] S.M. Samuels, The classical ruin problem with equal initial fortunes, Math. Mag. 48 (5) (1975) 286–288.
[3] W.A. Beyer, M.S. Waterman, Symmetries for conditioned ruin problems, Math. Mag. 50 (1) (1977) 42–45.
[4] William Feller, An Introduction to Probability Theory and its Applications. Vol. I, 3rd edition, John Wiley and Sons, Inc., New York, London, Sydney, 1968.
[5] K. Siegrist, n-point, win-by-k games, J. Appl. Probab. 26 (1989) 807–814.
[6] G.I. Bell, Models of carcinogenesis as an escape from mitotic inhibitors, Science 192 (4239) (1976) 569–572.
[7] Philippe Flajolet, Fabrice Guillemin, The formal theory of birth-and-death processes, lattice path combinatorics and continued fractions, Adv. Appl. Probab. 32 (3) (2000) 750–778.
[8] Ronald Graham, Donald Knuth, Oren Patashnik, Concrete Mathematics. A Foundation for Computer Science, Addison-Wesley Publishing Company, Reading, Massachusetts, 1989.
[9] E.W. Weisstein, CRC Concise Encyclopedia of Mathematics, 2nd edition, Chapman & Hall/CRC, 2002.
[10] Amy L. Rocha, Frederick Stern, The gambler's ruin problem with n players and asymmetric play, Statist. Probab. Lett. 44 (1) (1999) 87–95.
[11] A.L. Rocha, F. Stern, The asymmetric n-player gambler's ruin problem with equal initial fortunes, Adv. in Appl. Math. 33 (3) (2004) 512–530.

Please cite this article in press as: T. Lengyel, The conditional gambler's ruin problem with ties allowed, Appl. Math. Lett. (2008), doi:10.1016/j.aml.2008.04.007